

Black Holes in Supergravity and String Theory

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ABSTRACT

We give an elementary introduction to black holes in supergravity and string theory.² The focus is on the role of BPS solutions in four- and higher-dimensional supergravity and in string theory. Basic ideas and techniques are explained in detail, including exercises with solutions.

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1 Introduction

String theory has been the leading candidate for a unified quantum theory of all interactions during the last 15 years. The developments of the last five years have opened the possibility to go beyond perturbation theory and to address the most interesting problems of quantum gravity. Among the most prominent

of such problems are those related to black holes: the interpretation of the Bekenstein-Hawking entropy, Hawking radiation and the information problem.

The present set of lecture notes aims to give a paedagogical introduction to the subject of black holes in supergravity and string theory. It is primarily intended for graduate students who are interested in black hole physics, quantum gravity or string theory. No particular previous knowledge of these subjects is assumed, the notes should be accessible for any reader with some background in general relativity and quantum field theory. The basic ideas and techniques are treated in detail, including exercises and their solutions. This includes the definitions of mass, surface gravity and entropy of black holes, the laws of black hole mechanics, the interpretation of the extreme Reissner-Nordstrom black hole as a supersymmetric soliton, p -brane solutions of higher-dimensional supergravity, their interpretation in string theory and their relation to D -branes, dimensional reduction of supergravity actions, and, finally, the construction of extreme black holes by dimensional reduction of p -brane configurations. Other topics, which are needed to make the lectures self-contained are explained in a summaric way. Buscher T -duality is mentioned briefly and studied further in some of the exercises. Many other topics are omitted, according to the motto 'less is more'.

A short commented list of references is given at the end of every section. It is not intended to provide a representative or even full account of the literature, but to give suggestions for further reading. Therefore we recommend, based on subjective preference, some books, reviews and research papers.

2 Black holes in Einstein gravity

2.1 Einstein gravity

The basic idea of Einstein gravity is that the geometry of space-time is dynamical and is determined by the distribution of matter. Conversely the motion of matter is determined by the space-time geometry: In absence of non-gravitational forces matter moves along geodesics.

More precisely space-time is taken to be a (pseudo-) Riemannian manifold with metric $g_{\mu\nu}$. Our choice of signature is $(-+++)$. The reparametrization-invariant properties of the metric are encoded in the Riemann curvature tensor $R_{\mu\nu\rho\sigma}$, which is related by the gravitational field equations to the energy-momentum tensor of matter, $T_{\mu\nu}$. If one restricts the action to be at most quadratic in derivatives, and if one ignores the possibility of a cosmological constant,³ then the unique gravitational action is the Einstein-Hilbert action,

$$S_{EH} = \frac{1}{2\kappa^2} \int \sqrt{-g} R, \quad (2.1)$$

where κ is the gravitational constant, which will be related to Newton's constant below. The coupling to matter is determined by the principle of minimal cou-

³We will set the cosmological constant to zero throughout.

pling, i.e. one replaces partial derivatives by covariant derivatives with respect to the Christoffel connection $\Gamma_{\mu\nu}^\rho$.⁴

The energy-momentum tensor of matter is

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}}, \quad (2.2)$$

where S_M is the matter action. The Euler-Lagrange equations obtained from variation of the combined action $S_{EH} + S_M$ with respect to the metric are the Einstein equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa^2 T_{\mu\nu}. \quad (2.3)$$

Here $R_{\mu\nu}$ and R are the Ricci tensor and the Ricci scalar, respectively.

The motion of a massive point particle in a given space-time background is determined by the equation

$$ma^\nu = m\dot{x}^\mu \nabla_\mu \dot{x}^\nu = m(\ddot{x}^\nu + \Gamma_{\mu\rho}^\nu \dot{x}^\mu \dot{x}^\rho) = f^\nu, \quad (2.4)$$

where a^ν is the acceleration four-vector, f^ν is the force four-vector of non-gravitational forces and $\dot{x}^\mu = \frac{dx^\mu}{d\tau}$ is the derivative with respect to proper time τ .

In a flat background or in a local inertial frame equation (2.4) reduces to the force law of special relativity, $m\ddot{x}^\nu = f^\nu$. If no (non-gravitational) forces are present, equation (2.4) becomes the geodesic equation,

$$\dot{x}^\mu \nabla_\mu \dot{x}^\nu = \ddot{x}^\nu + \Gamma_{\mu\rho}^\nu \dot{x}^\mu \dot{x}^\rho = 0. \quad (2.5)$$

One can make contact with Newton gravity by considering the Newtonian limit. This is the limit of small curvature and non-relativistic velocities $v \ll 1$ (we take $c = \hbar = 1$). Then the metric can be expanded around the Minkowski metric

$$g_{\mu\nu} = \eta_{\mu\nu} + 2\psi_{\mu\nu}, \quad (2.6)$$

where $|\psi_{\mu\nu}| \ll 1$. If this expansion is carefully performed in the Einstein equation (2.3) and in the geodesic equation (2.5) one finds

$$\Delta V = 4\pi G_N \rho \quad \text{and} \quad \frac{d^2 \vec{x}}{dt^2} = -\vec{\nabla} V, \quad (2.7)$$

where V is the Newtonian potential, ρ is the matter density, which is the leading part of T_{00} , and G_N is Newton's constant. The proper time τ has been eliminated in terms of the coordinate time $t = x^0$. Thus one gets the potential equation for Newton's gravitational potential and the equation of motion for a point particle in it. The Newtonian potential V and Newton's constant G_N are related to ψ_{00} and κ by

$$V = -\psi_{00} \quad \text{and} \quad \kappa^2 = 8\pi G_N. \quad (2.8)$$

⁴In the case of fermionic matter one uses the vielbein e_μ^a instead of the metric and one introduces a second connection, the spin-connection ω_μ^{ab} , to which the fermions couple.

In Newtonian gravity a point mass or spherical mass distribution of total mass M gives rise to a potential $V = -G_N \frac{M}{r}$. According to (2.8) this corresponds to a leading order deformation of the flat metric of the form

$$g_{00} = -1 + 2 \frac{G_N M}{r} + O(r^{-2}) . \quad (2.9)$$

We will use equation (2.9) as our working definition for the mass of an asymptotically flat space-time. Note that there is no natural way to define the mass of a general space-time or of a space-time region. Although we have a local conservation law for the energy-momentum of matter, $\nabla^\mu T_{\mu\nu} = 0$, there is in general no way to construct a reparametrization invariant four-momentum by integration because $T_{\mu\nu}$ is a symmetric tensor. Difficulties in defining a meaningful conserved mass and four-momentum for a general space-time are also expected for a second reason. The principle of equivalence implies that the gravitational field can be eliminated locally by going to an inertial frame. Hence, there is no local energy density associated with gravity. But since the concept of mass works well in Newton gravity and in special relativity, we expect that one can define the mass of isolated systems, in particular the mass of an asymptotically flat space-time. Precise definitions can be given by different constructions, like the ADM mass and the Komar mass. More generally one can define the four-momentum and the angular momentum of an asymptotically flat space-time.

For practical purposes it is convenient to extract the mass by looking for the leading deviation of the metric from flat space, using (2.9). The quantity $r_S = 2G_N M$ appearing in the metric (2.9) has the dimension of a length and is called the Schwarzschild radius. From now on we will use Planckian units and set $G_N = 1$ on top of $\hbar = c = 1$, unless dimensional analysis is required.

2.2 The Schwarzschild black hole

Historically, the Schwarzschild solution was the first exact solution to Einstein's ever found. According to Birkhoff's theorem it is the unique spherically symmetric vacuum solution.

Vacuum solutions are those with a vanishing energy momentum tensor, $T_{\mu\nu} = 0$. By taking the trace of Einstein's equations this implies $R = 0$ and as a consequence

$$R_{\mu\nu} = 0 . \quad (2.10)$$

Thus the vacuum solutions to Einstein's equations are precisely the Ricci-flat space-times.

A metric is called spherically symmetric if it has a group of spacelike isometries with compact orbits which is isomorphic to the rotation group $SO(3)$. One can then go to adapted coordinates (t, r, θ, ϕ) , where t is time, r a radial variable and θ, ϕ are angular variables, such that the metric takes the form

$$ds^2 = -e^{2f(t,r)} dt^2 + e^{2g(t,r)} dr^2 + r^2 d\Omega^2 , \quad (2.11)$$

where $f(t, r), g(t, r)$ are arbitrary functions of t and r and $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$ is the line element on the unit two-sphere.

According to Birkhoff's theorem the Einstein equations determine the functions f, g uniquely. In particular such a solution must be static. A metric is called stationary if it has a timelike isometry. If one uses the integral lines of the corresponding Killing vector field to define the time coordinate t , then the metric is t -independent, $\partial_t g_{\mu\nu} = 0$. A stationary metric is called static if in addition the timelike Killing vector field is hypersurface orthogonal, which means that it is the normal vector field of a family of hypersurfaces. In this case one can eliminate the mixed components g_{ti} of the metric by a change of coordinates.⁵

In the case of a general spherically symmetric metric (2.11) the Einstein equations determine the functions f, g to be $e^{2f} = e^{-2g} = 1 - \frac{2M}{r}$. This is the Schwarzschild solution:

$$ds^2 = - \left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2. \quad (2.12)$$

Note that the solution is asymptotically flat, $g_{\mu\nu}(r) \rightarrow_{r \rightarrow \infty} \eta_{\mu\nu}$. According to the discussion of the last section, M is the mass of the Schwarzschild space-time.

One obvious feature of the Schwarzschild metric is that it becomes singular at the Schwarzschild radius $r_S = 2M$, where $g_{tt} = 0$ and $g_{rr} = \infty$. Before investigating this further let us note that r_S is very small: For the sun one finds $r_S = 2.9\text{km}$ and for the earth $r_S = 8.8\text{mm}$. Thus for atomic matter the Schwarzschild radius is inside the matter distribution. Since the Schwarzschild solution is a vacuum solution, it is only valid outside the matter distribution. Inside one has to find another solution with the energy-momentum tensor $T_{\mu\nu} \neq 0$ describing the system under consideration and one has to glue the two solutions at the boundary. The singularity of the Schwarzschild metric at r_S has no significance in this case. The same applies to nuclear matter, i.e. neutron stars. But stars with a mass above the Oppenheimer-Volkov limit of about 3 solar masses are unstable against total gravitational collapse. If such a collapse happens in a spherically symmetric way, then the final state must be the Schwarzschild metric, as a consequence of Birkhoff's theorem.⁶ In this situation the question of the singularity of the Schwarzschild metric at $r = r_S$ becomes physically relevant. As we will review next, $r = r_S$ is a so-called event horizon, and the solution describes a black hole. There is convincing observational evidence that such objects exist.

We now turn to the question what happens at $r = r_S$. One observation is that the singularity of the metric is a coordinate singularity, which can be

⁵In (2.11) these components have been eliminated using spherical symmetry.

⁶The assumption of a spherically symmetric collapse might seem unnatural. We will not discuss rotating black holes in these lecture notes, but there is a generalization of Birkhoff's theorem with the result that the most general uncharged stationary black hole solution in Einstein gravity is the Kerr black hole. A Kerr black hole is uniquely characterized by its mass and angular momentum. The stationary final state of an arbitrary collapse of neutral matter in Einstein gravity must be a Kerr black hole. Moreover rotating black holes, when interacting with their environment, rapidly lose angular momentum by superradiance. In the limit of vanishing angular momentum a Kerr black hole becomes a Schwarzschild black hole. Therefore even a non-spherical collapse of neutral matter can have a Schwarzschild black hole as its (classical) final state.

removed by going to new coordinates, for example to Eddington-Finkelstein or to Kruskal coordinates. As a consequence there is no curvature singularity, i.e. any coordinate invariant quantity formed out of the Riemann curvature tensor is finite. In particular the tidal forces on any observer at $r = r_S$ are finite and even arbitrarily small if one makes r_S sufficiently large. Nevertheless the surface $r = r_S$ is physically distinguished: It is a future event horizon. This property can be characterized in various ways.

Consider first the free radial motion of a massive particle (or of a local observer in a starship) between positions $r_2 > r_1$. Then the time $\Delta t = t_1 - t_2$ needed to travel from r_2 to r_1 diverges in the limit $r_1 \rightarrow r_S$:

$$\Delta t \simeq r_S \log \frac{r_2 - r_S}{r_1 - r_S} \xrightarrow{r_1 \rightarrow r_S} \infty. \quad (2.13)$$

Does this mean that one cannot reach the horizon? Here we have to remember that the time t is the coordinate time, i.e. a timelike coordinate that we use to label events. It is not identical with the time measured by a freely falling observer. Since the metric is asymptotically flat, the Schwarzschild coordinate time coincides with the proper time of an observer at rest at infinity. Loosely speaking an observer at infinity (read: far away from the black hole) never 'sees' anything reach the horizon. This is different from the perspective of a freely falling observer. For him the difference $\Delta\tau = \tau_1 - \tau_2$ of proper time is finite:

$$\Delta\tau = \tau_1 - \tau_2 = \frac{2}{3\sqrt{3}r_S} \left(r_2^{3/2} - r_1^{3/2} \right) \xrightarrow{r_1 \rightarrow r_S} \frac{2}{3\sqrt{3}r_S} \left(r_2^{3/2} - r_S^{3/2} \right). \quad (2.14)$$

As discussed above the gravitational forces at r_S are finite and the freely falling observer will enter the interior region $r < r_S$. The consequences will be considered below.

Obviously the proper time of the freely falling observer differs the more from the Schwarzschild time the closer he gets to the horizon. The precise relation between the infinitesimal time intervals is

$$\frac{d\tau}{dt} = \sqrt{-g_{tt}} = \left(1 - \frac{r_S}{r} \right)^{1/2} =: V(r). \quad (2.15)$$

The quantity $V(r)$ is called the redshift factor associated with the position r . This name is motivated by our second thought experiment. Consider two static observers at positions $r_1 < r_2$. The observer at r_1 emits a light ray of frequency ω_1 which is registered at r_2 with frequency ω_2 . The frequencies are related by

$$\frac{\omega_1}{\omega_2} = \frac{V(r_2)}{V(r_1)}. \quad (2.16)$$

Since $\frac{V(r_2)}{V(r_1)} < 1$, a lightray which travels outwards is redshifted, $\omega_2 < \omega_1$. Moreover, since the redshift factor vanishes at the horizon, $V(r_1 = r_S) = 0$, the frequency ω_2 goes to zero, if the source is moved to the horizon. Thus, the event horizon can be characterized as a surface of infinite redshift.

Exercise I : Compute the Schwarzschild time that a lightray needs in order to travel from r_1 to r_2 . What happens in the limit $r_1 \rightarrow r_S$?

Exercise II : Derive equation (2.16).

Hint 1: If k_μ is the four-momentum of the lightray and if u_i^μ is the four-velocity of the static observer at r_i , $i = 1, 2$, then the frequency measured in the frame of the static observer is

$$\omega_i = -k_\mu u_i^\mu . \quad (2.17)$$

(why is this true?).

Hint 2: If ξ^μ is a Killing vector field and if t^μ is the tangent vector to a geodesic, then

$$t^\mu \nabla_\mu (\xi_\nu k^\nu) = 0 , \quad (2.18)$$

i.e. there is a conserved quantity. (Proof this. What is the meaning of the conserved quantity?)

Hint 3: What is the relation between ξ^μ and u_i^μ ?

Finally, let us give a third characterization of the event horizon. This will also enable us to introduce a quantity called the surface gravity, which will play an important role later. Consider a static observer at position $r > r_S$ in the Schwarzschild space-time. The corresponding world line is not a geodesic and therefore there is a non-vanishing acceleration a^μ . In order to keep a particle (or starship) of mass m at position, a non-gravitational force $f^\mu = ma^\mu$ must act according to (2.4). For a Schwarzschild space-time the acceleration is computed to be

$$a^\mu = \nabla^\mu \log V(r) \quad (2.19)$$

and its absolute value is

$$a = \sqrt{a^\mu a_\mu} = \frac{\sqrt{\nabla_\mu V(r) \nabla^\mu V(r)}}{V(r)} . \quad (2.20)$$

Whereas the numerator is finite at the horizon

$$\sqrt{\nabla_\mu V(r) \nabla^\mu V(r)} = \frac{r_S}{2r^2} \rightarrow_{r \rightarrow r_S} \frac{1}{2r_S} , \quad (2.21)$$

the denominator, which is just the redshift factor, goes to zero and the acceleration diverges. Thus the event horizon is a place where one cannot keep position. The finite quantity

$$\kappa_S := (Va)_{r=r_S} \quad (2.22)$$

is called the surface gravity of the event horizon. This quantity characterizes the strength of the gravitational field. For a Schwarzschild black hole we find

$$\kappa_S = \frac{1}{2r_S} = \frac{1}{4M} . \quad (2.23)$$

Exercise III : Derive (2.19), (2.20) and (2.23).

Summarizing we have found that the interior region $r < r_S$ can be reached in finite proper time from the exterior but is causally decoupled in the sense that no matter or light can get back from the interior to the exterior region. The future event horizon acts like a semipermeable membrane which can only be crossed from outside to inside.⁷

Let us now briefly discuss what happens in the interior region. The proper way to proceed is to introduce new coordinates, which are regular at $r = r_S$ and then to analytically continue to $r < r_S$. Examples of such coordinates are Eddington-Finkelstein or Kruskal coordinates. But it turns out that the interior region $0 < r < r_S$ of the Schwarzschild metric (2.12) is isometric to the corresponding region of the analytically continued metric. Thus we might as well look at the Schwarzschild metric at $0 < r < r_S$. And what we see is suggestive: the terms g_{tt} and g_{rr} in the metric flip sign, which says that 'time' t and 'space' r exchange their roles.⁸ In the interior region r is a timelike coordinate and every timelike or lightlike geodesic has to proceed to smaller and smaller values of r until it reaches the point $r = 0$. One can show that every timelike geodesic reaches this point in finite proper time (whereas lightlike geodesics reach it at finite 'affine parameter', which is the substitute of proper time for light rays).

Finally we have to see what happens at $r = 0$. The metric becomes singular but this time the curvature scalar diverges, which shows that there is a curvature singularity. Extended objects are subject to infinite tidal forces when reaching $r = 0$. It is not possible to analytically continue geodesics beyond this point.

2.3 The Reissner-Nordstrom black hole

We now turn our attention to Einstein-Maxwell theory. The action is

$$S = \int d^4x \sqrt{-g} \left(\frac{1}{2\kappa^2} R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right). \quad (2.24)$$

The curved-space Maxwell equations are the combined set of the Euler-Lagrange equations and Bianchi identities for the gauge fields:

$$\nabla_\mu F^{\mu\nu} = 0, \quad (2.25)$$

$$\varepsilon^{\mu\nu\rho\sigma} \partial_\nu F_{\rho\sigma} = 0. \quad (2.26)$$

⁷In the opposite case one would call it a past event horizon and the corresponding space-time a white hole.

⁸Actually the situation is slightly asymmetric between t and r . r is a good coordinate both in the exterior region $r > r_S$ and interior region $r < r_S$. On the other hand t is a coordinate in the exterior region, and takes its full range of values $-\infty < t < \infty$ there. The associated timelike Killing vector field becomes lightlike on the horizon and spacelike in the interior. One can introduce a spacelike coordinate using its integral lines, and if one calls this coordinate t , then the metric takes the form of a Schwarzschild metric with $r < r_S$. But note that the 'interior t ' is not the analytic extension of the Schwarzschild time, whereas r has been extended analytically to the interior.

Introducing the dual gauge field

$${}^*F_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma} , \quad (2.27)$$

one can rewrite the Maxwell equations in a more symmetric way, either as

$$\nabla_\mu F^{\mu\nu} = 0 \text{ and } \nabla_\mu {}^*F^{\mu\nu} = 0 \quad (2.28)$$

or as

$$\varepsilon^{\mu\nu\rho\sigma}\partial_\nu {}^*F_{\rho\sigma} = 0 \text{ and } \varepsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = 0 . \quad (2.29)$$

In this form it is obvious that the Maxwell equations are invariant under duality transformations

$$\begin{pmatrix} F_{\mu\nu} \\ {}^*F_{\mu\nu} \end{pmatrix} \longrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} F_{\mu\nu} \\ {}^*F_{\mu\nu} \end{pmatrix} , \text{ where } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{R}) . \quad (2.30)$$

These transformations include electric-magnetic duality transformations $F_{\mu\nu} \rightarrow {}^*F_{\mu\nu}$. Note that duality transformations are invariances of the field equations but not of the action.

In the presence of source terms the Maxwell equations are no longer invariant under continuous duality transformations. If both electric and magnetic charges exist, one can still have an invariance. But according to the Dirac quantization condition the spectrum of electric and magnetic charges is discrete and the duality group is reduced to a discrete subgroup of $GL(2, \mathbb{R})$.

Electric and magnetic charges q, p can be written as surface integrals,

$$q = \frac{1}{4\pi} \oint {}^*F , \quad p = \frac{1}{4\pi} \oint F , \quad (2.31)$$

where $F = \frac{1}{2}F_{\mu\nu}dx^\mu dx^\nu$ is the field strength two-form and the integration surface surrounds the sources. Note that the integrals have a reparametrization invariant meaning because one integrates a two-form. This was different for the mass.

Exercise IV : Solve the Maxwell equations in a static and spherically symmetric background,

$$ds^2 = -e^{2g(r)}dt^2 + e^{2f(r)}dr^2 + r^2 d\Omega^2 \quad (2.32)$$

for a static and spherically symmetric gauge field.

We now turn to the gravitational field equations,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa^2 \left(F_{\mu\rho}F^\rho{}_\nu - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} \right) . \quad (2.33)$$

Taking the trace we get $R = 0$. This is always the case if the energy-momentum tensor is traceless.

There is a generalization of Birkhoff's theorem: The unique spherically symmetric solution of (2.33) is the Reissner-Nordstrom solution

$$\begin{aligned} ds^2 &= -e^{2f(r)} dt^2 + e^{-2f(r)} dr^2 + r^2 d\Omega^2 \\ F_{tr} &= -\frac{q}{r^2}, \quad F_{\theta\phi} = p \sin \theta \\ e^{2f(r)} &= 1 - \frac{2M}{r} + \frac{q^2 + p^2}{r^2} \end{aligned} \quad (2.34)$$

where M, q, p are the mass and the electric and magnetic charge. The solution is static and asymptotically flat.

Exercise V : Show that q, p are the electric and magnetic charge, as defined in (2.31).

Exercise VI : Why do the electro-static field F_{tr} and the magneto-static field $F_{\theta\phi}$ look so different?

Note that it is sufficient to know the electric Reissner-Nordstrom solution, $p = 0$. The dyonic generalization can be generated by a duality transformation.

We now have to discuss the Reissner-Nordstrom metric. It is convenient to rewrite

$$e^{2f} = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} = \left(1 - \frac{r_+}{r}\right) \left(1 - \frac{r_-}{r}\right), \quad (2.35)$$

where we set $Q = \sqrt{q^2 + p^2}$ and

$$r_{\pm} = M \pm \sqrt{M^2 - Q^2}. \quad (2.36)$$

There are three cases to be distinguished:

1. $M > Q > 0$: The solution has two horizons, an event horizon at r_+ and a so-called Cauchy horizon at r_- . This is the non-extreme Reissner-Nordstrom black hole. The surface gravity is $\kappa_S = \frac{r_+ - r_-}{2r_+^2}$.
2. $M = Q > 0$: In this limit the two horizons coincide at $r_+ = r_- = M$ and the mass equals the charge. This is the extreme Reissner-Nordstrom black hole. The surface gravity vanishes, $\kappa_S = 0$.
3. $M < Q$: There is no horizon and the solution has a naked singularity. Such solutions are believed to be unphysical. According to the cosmic censorship hypothesis the only physical singularities are the big bang, the big crunch, and singularities hidden behind event horizons, i.e. black holes.

2.4 The laws of black hole mechanics

We will now discuss the laws of black hole mechanics. This is a remarkable set of relations, which is formally equivalent to the laws of thermodynamics. The significance of this will be discussed later. Before we can formulate the laws, we need a few definitions.

First we need to give a general definition of a black hole and of a (future) event horizon. Intuitively a black hole is a region of space-time from which one cannot escape. In order to make the term 'escape' more precise, one considers the behaviour of time-like geodesics. In Minkowski space all such curves have the same asymptotics. Since the causal structure is invariant under conformal transformations, one can describe this by mapping Minkowski space to a finite region and adding 'points at infinity'. This is called a Penrose diagram. In Minkowski space all timelike geodesics end at the same point, which is called 'future timelike infinity'. The backward lightcone of this point is all of Minkowski space. If a general space-time contains an asymptotically flat region, one can likewise introduce a point at future timelike infinity. But it might happen that its backward light cone is not the whole space. In this case the space-time contains timelike geodesics which do not 'escape' to infinity. The region which is not in the backward light cone of future timelike infinity is a black hole or a collection of black holes. The boundary of the region of no-escape is called a future event horizon. By definition it is a lightlike surface, i.e. its normal vector field is lightlike.

In Einstein gravity the event horizons of stationary black holes are so-called Killing horizons. This property is crucial for the derivation of the zeroth and first law. A Killing horizon is defined to be a lightlike hypersurface where a Killing vector field becomes lightlike. For static black holes in Einstein gravity the horizon Killing vector field is $\xi = \frac{\partial}{\partial t}$. Stationary black holes in Einstein gravity are axisymmetric and the horizon Killing vector field is

$$\xi = \frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \phi}, \quad (2.37)$$

where Ω is the rotation velocity and $\frac{\partial}{\partial \phi}$ is the Killing vector field of the axial symmetry.

The zeroth and first law do not depend on particular details of the gravitational field equations. They can be derived in higher derivative gravity as well, provided one makes the following assumptions, which in Einstein gravity follow from the field equations: One has to assume that (i) the event horizon is a Killing horizon and (ii) that the black hole is either static or that it is stationary, axisymmetric and possesses a discrete $t - \phi$ reflection symmetry.⁹

For a Killing horizon one can define the surface gravity κ_S by the equation

$$\nabla_\mu (\xi^\nu \xi_\nu) = -2\kappa_S \xi_\mu, \quad (2.38)$$

⁹ This means that in adapted coordinates (t, ϕ, \dots) the $g_{t\phi}$ -component of the metric vanishes.

which is valid on the horizon. The meaning of this equation is as follows: The Killing horizon is defined by the equation $\xi^\nu \xi_\nu = 0$. The gradient of the defining equation of a surface is a normal vector field to the surface. Since ξ_μ is also a normal vector field both have to be proportional. The factor between the two vectors fields defines the surface gravity by (2.38). A priori the surface gravity is a function on the horizon. But according to the zeroth law of black hole mechanics it is actually a constant,

$$\kappa_S = \text{const.} \quad (2.39)$$

The first law of black hole mechanics is energy conservation: when comparing two infinitesimally close stationary black holes in Einstein gravity one finds:

$$\delta M = \frac{1}{8\pi} \kappa_S \delta A + \Omega \delta J + \mu \delta Q . \quad (2.40)$$

Here A denotes the area of the event horizon, J is the angular momentum and Q the charge. Ω is the rotation velocity and $\mu = \frac{Qr_+}{A}$.

The comparison of the zeroth and first law of black hole mechanics to the zeroth and first law thermodynamics,

$$T = \text{const} , \quad (2.41)$$

$$\delta E = T \delta S + p \delta V + \mu \delta N , \quad (2.42)$$

suggests to identify surface gravity with temperature and the area of the event horizon with entropy:

$$\kappa_S \sim T , \quad A \sim S . \quad (2.43)$$

Classically this identification does not seem to have physical content, because a black hole cannot emit radiation and therefore has temperature zero. This changes when quantum mechanics is taken into account: A stationary black hole emits Hawking radiation, which is found to be proportional to its surface gravity:

$$T_H = \frac{\kappa_S}{2\pi} . \quad (2.44)$$

This fixes the factor between area and entropy:

$$S_{BH} = \frac{A}{4} \frac{1}{G_N} . \quad (2.45)$$

In this formula we reintroduced Newton's constant in order to show that the black hole entropy is indeed dimensionless (we have set the Boltzmann constant to unity). The relation (2.45) is known as the area law and S_{BH} is called the Bekenstein-Hawking entropy. The Hawking effect shows that it makes sense to identify κ_S with the temperature, but can we show directly that S_{BH} is the entropy? And where does the entropy of a black hole come from?

We are used to think about entropy in terms of statistical mechanics. Systems with a large number of degrees of freedom are conveniently described using

two levels of description: A microscopic description where one uses all degrees of freedom and a coarse-grained, macroscopic description where one uses a few observables which characterize the interesting properties of the system. In the case of black holes we know a macroscopic description in terms of classical gravity. The macroscopic observables are the mass M , the angular momentum J and the charge Q , whereas the Bekenstein-Hawking entropy plays the role of the thermodynamic entropy. What is lacking so far is a microscopic level of description. For certain extreme black holes we will discuss a proposal of such a description in terms of D -branes later. Assuming that we have a microscopic description the microscopic or statistical entropy is

$$S_{\text{stat}} = \log N(M, Q, J) , \quad (2.46)$$

where $N(M, Q, J)$ is the number of microstates which belong to the same macrostate. If the interpretation of S_{BH} as entropy is correct, then the macroscopic and microscopic entropies must coincide:

$$S_{BH} = S_{\text{stat}} . \quad (2.47)$$

We will see later that this is indeed true for the D -brane picture of black holes.

2.5 Literature

Our discussion of gravity and black holes and most of the exercises follow the book by Wald [1], which we recommend for further study. The two monographies [2] and [3] cover various aspects of black hole physics in great detail.

3 Black holes in supergravity

We now turn to the discussion of black holes in the supersymmetric extension of gravity, called supergravity. The reason for this is two-fold. The first is that we want to discuss black holes in the context of superstring theory, which has supergravity as its low energy limit. The second reason is that extreme black holes are supersymmetric solitons. As a consequence quantum corrections are highly constrained and this can be used to make quantitative tests of the microscopic D -brane picture of black holes.

3.1 The extreme Reissner-Nordstrom black hole

Before discussing supersymmetry we will collect several special properties of extreme Reissner-Nordstrom black holes. These will be explained in terms of supersymmetry later.

The metric of the extreme Reissner-Nordstrom black hole is

$$ds^2 = - \left(1 - \frac{M}{r}\right)^2 dt^2 + \left(1 - \frac{M}{r}\right)^{-2} dr^2 + r^2 d\Omega^2 , \quad (3.1)$$

where $M = \sqrt{q^2 + p^2}$. By a coordinate transformation one can make the spatial part of the metric conformally flat. Such coordinates are called isotropic:

$$ds^2 = - \left(1 + \frac{M}{r}\right)^{-2} dt^2 + \left(1 + \frac{M}{r}\right)^2 (dr^2 + r^2 d\Omega^2) . \quad (3.2)$$

Note that the new coordinates only cover the region outside the horizon, which now is located at $r = 0$.

The isotropic form of the metric is useful for exploring its special properties. In the near horizon limit $r \rightarrow 0$ we find

$$ds^2 = -\frac{r^2}{M^2} dt^2 + \frac{M^2}{r^2} dr^2 + M^2 d\Omega^2 . \quad (3.3)$$

The metric factorizes asymptotically into two two-dimensional spaces, which are parametrized by (t, r) and (θ, ϕ) , respectively. The (θ, ϕ) -space is obviously a two-sphere of radius M , whereas the (t, r) -space is the two-dimensional Anti-de Sitter space AdS^2 , with radius M . Both are maximally symmetric spaces:

$$S^2 = \frac{SO(3)}{SO(2)}, \quad AdS^2 = \frac{SO(2, 1)}{SO(1, 1)} . \quad (3.4)$$

The scalar curvatures of the two factors are proportional to $\pm M^{-1}$ and precisely cancel, as they must, because the product space has a vanishing curvature scalar, $R = 0$, as a consequence of $T^\mu_\mu = 0$.

The $AdS^2 \times S^2$ space is known as the Bertotti-Robinson solution. More precisely it is one particular specimen of the family of Bertotti-Robinson solutions, which are solutions of Einstein-Maxwell theory with covariantly constant electromagnetic field strength. The particular solution found here corresponds to the case with vanishing cosmological constant and absence of charged matter.

The metric (3.3) has one more special property: it is conformally flat.

Exercise VII : Find the coordinate transformation that maps (3.1) to (3.2). Show that in isotropic coordinates the 'point' $r = 0$ is a sphere of radius M and area $A = 4\pi M^2$. Show that the metric (3.3) is conformally flat. (Hint: It is not necessary to compute the Weyl curvature tensor. Instead, there is a simple coordinate transformation which makes conformal flatness manifest.)

We next discuss another astonishing property of the extreme Reissner-Nordstrom solution. Let us drop spherical symmetry and look for solutions of Einstein-Maxwell theory with a metric of the form

$$ds^2 = -e^{-2f(\vec{x})} dt^2 + e^{2f(\vec{x})} d\vec{x}^2 . \quad (3.5)$$

In such a background the Maxwell equations are solved by electrostatic fields with a potential given in terms of $f(\vec{x})$:

$$F_{ti} = \mp \partial_i(e^{-f}) , \quad F_{ij} = 0 . \quad (3.6)$$

More general dyonic solutions which carry both electric and magnetic charge can be generated by duality transformations. The only constraint that the coupled Einstein and Maxwell equations impose on f is that e^f must be a harmonic function,

$$\Delta e^f = \sum_{i=1}^3 \partial_i \partial_i e^f = 0. \quad (3.7)$$

Note that Δ is the flat Laplacian. The solution (3.5,3.6,3.7) is known as the Majumdar-Papapetrou solution.

Exercise VIII : Show that (3.6) solves the Maxwell equations in the metric background (3.5) if and only if e^f is harmonic.

One possible choice of the harmonic function is

$$e^f = 1 + \frac{M}{r}. \quad (3.8)$$

This so-called single-center solution is the extreme Reissner-Nordstrom black hole with mass $M = \sqrt{q^2 + p^2}$.

The more general harmonic function

$$e^f = 1 + \sum_{I=1}^N \frac{M_I}{|\vec{x} - \vec{x}_I|} \quad (3.9)$$

is a so-called multi-center solution, which describes a static configuration of extreme Reissner-Nordstrom black holes with horizons located at positions \vec{x}_I .

These positions are completely arbitrary: gravitational attraction and electrostatic and magnetostatic repulsion cancel for every choice of \vec{x}_I . This is called the no-force property.

The masses of the black holes are

$$M_I = \sqrt{q_I^2 + p_I^2}, \quad (3.10)$$

where q_I, p_I are the electric and magnetic charges. For purely electric solutions, $p_I = 0$, the Maxwell equations imply that $\pm q_I = M_I$, depending on the choice of sign in (3.6). In order to avoid naked singularities we have to take all the masses to be positive. As a consequence either all the charges q_I are positive or they are negative. This is natural, because one needs to cancel the gravitational attraction by electrostatic repulsion in order to have a static solution. In the case of a dyonic solution all the complex numbers $q_I + ip_I$ must have the same phase in the complex plane.

Finally one might ask whether other choices of the harmonic function yield interesting solutions. The answer is no, because all other choices lead to naked singularities.

Let us then collect the special properties of the extreme Reissner-Nordstrom black hole: It saturates the mass bound for the presence of an event horizon

and has vanishing surface gravity and therefore vanishing Hawking temperature. The solution interpolates between two maximally symmetric geometries: Flat space at infinity and the Bertotti-Robinson solution at the horizon. Finally there exist static multi-center solutions with the remarkable no-force property.

As usual in physics special properties are expected to be manifestations of a symmetry. We will now explain that the symmetry is (extended) supersymmetry. Moreover the interpolation property and the no-force property are reminiscent of the Prasad Sommerfield limit of 't Hooft Polyakov monopoles in Yang-Mills theory. This is not a coincidence: The extreme Reissner-Nordstrom is a supersymmetric soliton of extended supergravity.

3.2 Extended supersymmetry

We will now review the supersymmetry algebra and its representations. Supersymmetric theories are theories with conserved spinorial currents. If N such currents are present, one gets $4N$ real conserved charges, which can either be organized into N Majorana spinors Q_m^A or into N Weyl spinors Q_α^A . Here $A = 1, \dots, N$ counts the supersymmetries, whereas $m = 1, \dots, 4$ is a Majorana spinor index and $\alpha = 1, 2$ is a Weyl spinor index. The hermitean conjugate of Q_α^A is denoted by Q_α^{+A} . It has opposite chirality, but we refrain from using dotted indices.

According to the theorem of Haag, Lopuzanski and Sohnius the most general supersymmetry algebra (in four space-time dimensions) is

$$\{Q_\alpha^A, Q_\beta^{+B}\} = 2\sigma_{\alpha\beta}^\mu P_\mu \delta^{AB}, \quad (3.11)$$

$$\{Q_\alpha^A, Q_\beta^B\} = 2\varepsilon_{\alpha\beta} Z^{AB}. \quad (3.12)$$

In the case of extended supersymmetry, $N > 1$, not only the momentum operator P_μ , but also the operators Z^{AB} occur on the right hand side of the anticommutation relations. The matrix Z^{AB} is antisymmetric. The operators in Z^{AB} commute with all operators in the super Poincaré algebra and therefore they are called central charges. In the absence of central charges the automorphism group of the algebra is $U(N)$. If central charges are present the automorphism group is reduced to $USp(2N) = U(N) \cap Sp(2N, \mathbb{C})$.¹⁰ One can then use $U(N)$ transformations which are not symplectic to skew-diagonalize the antisymmetric matrix Z^{AB} .

For concreteness we now specialize to the case $N = 2$. We want to construct representations and we start with massive representations, $M^2 > 0$. Then the momentum operator can be brought to the standard form $P_\mu = (-M, \vec{0})$. Plugging this into the algebra and setting $2|Z| = |Z^{12}|$ the algebra takes the form

$$\{Q_\alpha^A, Q_\beta^{+B}\} = 2M\delta_{\alpha\beta}\delta^{AB},$$

¹⁰Our convention concerning the symplectic group is that $Sp(2)$ has rank 1. In other words the argument is always even.

$$\{Q_\alpha^A, Q_\beta^B\} = 2|Z|\varepsilon_{\alpha\beta}\varepsilon^{AB}. \quad (3.13)$$

The next step is to rewrite the algebra using fermionic creation and annihilation operators. By taking appropriate linear combinations of the supersymmetry charges one can bring the algebra to the form

$$\begin{aligned} \{a_\alpha, a_\beta^+\} &= 2(M + |Z|)\delta_{\alpha\beta}, \\ \{b_\alpha, b_\beta^+\} &= 2(M - |Z|)\delta_{\alpha\beta}. \end{aligned} \quad (3.14)$$

Now one can choose any irreducible representation $[s]$ of the little group $SO(3)$ of massive particles and take the a_α, b_β to be annihilation operators,

$$a_\alpha|s\rangle = 0, \quad b_\beta|s\rangle = 0. \quad (3.15)$$

Then the basis of the corresponding irreducible representation of the super Poincaré algebra is

$$\mathcal{B} = \{a_{\alpha_1}^+ \cdots b_{\beta_1}^+ \cdots |s\rangle\}. \quad (3.16)$$

In the context of quantum mechanics we are only interested in unitary representations. Therefore we have to require the absence of negative norm states. This implies that the mass is bounded by the central charge:

$$M \geq |Z|. \quad (3.17)$$

This is called the BPS-bound, a term originally coined in the context of monopoles in Yang-Mills theory. The representations fall into two classes. If $M > |Z|$, then we immediately get unitary representations. Since we have 4 creation operators the dimension is $2^4 \cdot \dim[s]$. These are the so-called long representations. The most simple example is the long vector multiplet with spin content $(1[1], 4[\frac{1}{2}], 5[0])$. It has 8 bosonic and 8 fermionic on-shell degrees of freedom.

If the BPS bound is saturated, $M = |Z|$, then the representation contains null states, which have to be divided out in order to get a unitary representation. This amounts to setting the b -operators to zero. As a consequence half of the supertransformations act trivially. This is usually phrased as: The multiplet is invariant under half of the supertransformations. The basis of the unitary representation is

$$\mathcal{B}' = \{a_{\alpha_1}^+ \cdots |s\rangle\}. \quad (3.18)$$

Since there are only two creation operators, the dimension is $2^2 \cdot \dim[s]$. These are the so-called short representations or BPS representations. Note that the relation $M = |Z|$ is a consequence of the supersymmetry algebra and therefore cannot be spoiled by quantum corrections (assuming that the full theory is supersymmetric).

There are two important examples of short multiplets. One is the short vector multiplet, with spin content $(1[1], 2[\frac{1}{2}], 1[0])$, the other is the hypermultiplet

with spin content $(2[\frac{1}{2}], 4[0])$. Both have four bosonic and four fermionic on-shell degrees of freedom.

Let us also briefly discuss massless representations. In this case the momentum operator can be brought to the standard form $P_\mu = (-E, 0, 0, E)$ and the little group is $ISO(2)$, the two-dimensional Euclidean group. Irreducible representations of the Poincaré group are labeled by their helicity h , which is the quantum number of the representation of the subgroup $SO(2) \subset ISO(2)$. Similar to short representations one has to set half of the operators to zero in order to obtain unitary representations. Irreducible representations of the super Poincaré group are obtained by acting with the remaining two creation operators on a helicity eigenstate $|h\rangle$. Note that the resulting multiplets will in general not be CP selfconjugate. Thus one has to add the CP conjugated multiplet to describe the corresponding antiparticles. There are three important examples of massless $N = 2$ multiplets. The first is the supergravity multiplet with helicity content $(1[\pm 2], 2[\pm \frac{3}{2}], 1[\pm 1])$. The states correspond to the graviton, two real gravitini and a gauge boson, called the graviphoton. The bosonic field content is precisely the one of Einstein-Maxwell theory. Therefore Einstein-Maxwell theory can be embedded into $N = 2$ supergravity by adding two gravitini. The other two important examples of massless multiplets are the massless vector and hypermultiplet, which are massless versions of the corresponding massive short multiplets.

In supersymmetric field theories the supersymmetry algebra is realized as a symmetry acting on the underlying fields. The operator generating an infinitesimal supertransformation takes the form $\delta_\epsilon^Q = \bar{\epsilon}_A^m Q_m^A$, when using Majorana spinors. The transformation parameters ϵ_A^m are N anticommuting Majorana spinors. Depending on whether they are constant or space-time dependent, supersymmetry is realized as a rigid or local symmetry, respectively. In the local case, the anticommutator of two supertransformations yields a local translation, i.e. a general coordinate transformation. Therefore locally supersymmetric field theories have to be coupled to a supersymmetric extension of gravity, called supergravity. The gauge fields of general coordinate transformations and of local supertransformations are the graviton, described by the vielbein e_μ^a and the gravitini $\psi_\mu^A = \psi_{\mu m}^A$. They sit in the supergravity multiplet. We have specified the $N = 2$ supergravity multiplet above.

We will now explain why we call the extreme Reissner-Nordstrom black hole a 'supersymmetric soliton'. Solitons are stationary, regular and stable finite energy solutions to the equations of motion. The extreme Reissner-Nordstrom black hole is stationary (even static) and has finite energy (mass). It is regular in the sense of not having a naked singularity. We will argue below that it is stable, at least when considered as a solution of $N = 2$ supergravity. What do we mean by a 'supersymmetric' soliton? Generic solutions to the equations of motion will not preserve any of the symmetries of the vacuum. In the context of gravity space-time symmetries are generated by Killing vectors. The trivial vacuum, Minkowski space, has ten Killing vectors, because it is Poincaré invariant. A generic space-time will not have any Killing vectors, whereas special,

more symmetric space-times have some Killing vectors, but not the maximal number of 10. For example the Reissner-Nordstrom black hole has one time-like Killing vector field corresponding to time translation invariance and three spacelike Killing vector fields corresponding to rotation invariance. But the spatial translation invariance is broken, as it must be for a finite energy field configuration. Since the underlying theory is translation invariant, all black hole solutions related by rigid translations are equivalent and have in particular the same energy. In this way every symmetry of the vacuum which is broken by the field configuration gives rise to a collective mode.

Similarly a solution is called supersymmetric if it is invariant under a rigid supertransformation. In the context of locally supersymmetric theories such residual rigid supersymmetries are the fermionic analogues of isometries. A field configuration Φ_0 is supersymmetric if there exists a choice $\epsilon(x)$ of the supersymmetry transformation parameters such that the configuration is invariant,

$$\delta_{\epsilon(x)}\Phi|_{\Phi_0} = 0 . \quad (3.19)$$

As indicated by notation one has to perform a supersymmetry variation of all the fields Φ , with parameter $\epsilon(x)$ and then to evaluate it on the field configuration Φ_0 . The transformation parameters $\epsilon(x)$ are fermionic analogues of Killing vectors and therefore they are called Killing spinors. Equation (3.19) is referred to as the Killing spinor equation. As a consequence of the residual supersymmetry the number of fermionic collective modes is reduced. If the solution is particle like, i.e. asymptotically flat and of finite mass, then we expect that it sits in a short multiplet and describes a BPS state of the theory.

Let us now come back to the extreme Reissner-Nordstrom black hole. This is a solution of Einstein-Maxwell theory, which can be embedded into $N = 2$ supergravity by adding two gravitini ψ_μ^A . The extreme Reissner-Nordstrom black hole is also a solution of the extended theory, with $\psi_\mu^A = 0$. Moreover it is a supersymmetric solution in the above sense, i.e. it possesses Killing spinors. What are the Killing spinor equations in this case? The graviton e_μ^a and the graviphoton A_μ transform into fermionic quantities, which all vanish when evaluated in the background. Hence the only conditions come from the gravitino variation:

$$\delta_\epsilon \psi_{\mu A} = \nabla_\mu \epsilon_A - \frac{1}{4} F_{ab}^- \gamma^a \gamma^b \gamma_\mu \epsilon_{AB} \epsilon^B \stackrel{!}{=} 0 . \quad (3.20)$$

The notation and conventions used in this equation are as follows: We suppress all spinor indices and use the so-called chiral notation. This means that we use four-component Majorana spinors, but project onto one chirality, which is encoded in the position of the supersymmetry index $A = 1, 2$:

$$\gamma_5 \epsilon^A = \epsilon^A \quad \gamma_5 \epsilon_A = -\epsilon_A . \quad (3.21)$$

As a consequence of the Majorana condition only half of the components of ϵ_A, ϵ^A are independent, i.e. there are 8 real supertransformation parameters. The indices μ, ν are curved and the indices a, b are flat tensor indices. $F_{\mu\nu}$ is the graviphoton field strength and

$$F_{\mu\nu}^\pm = \frac{1}{2} (F_{\mu\nu} \pm i {}^* F_{\mu\nu}) \quad (3.22)$$

are its selfdual and antiselfdual part.

One can now check that the Majumdar-Papapetrou solution and in particular the extreme Reissner-Nordstrom black hole have Killing spinors

$$\epsilon_A(\vec{x}) = h(\vec{x})\epsilon_A(\infty) , \quad (3.23)$$

where $h(\vec{x})$ is completely fixed in terms of $f(\vec{x})$. The values of the Killing spinors at infinity are subject to the condition

$$\epsilon_A(\infty) + i\gamma^0 \frac{Z}{|Z|} \varepsilon_{AB} \epsilon^B(\infty) = 0 . \quad (3.24)$$

This projection fixes half of the parameters in terms of the others. As a consequence we have four Killing spinors, which is half of the maximal number eight. The four supertransformations which do not act trivially correspond to four fermionic collective modes. It can be shown that the extreme Reissner-Nordstrom black hole is part of a hypermultiplet. The quantity Z appearing in the phase factor $Z/|Z|$ is the central charge. In locally supersymmetric theories the central charge transformations are local $U(1)$ transformations, and the corresponding gauge field is the graviphoton. The central charge is a complex linear combination of the electric and magnetic charge of this $U(1)$:

$$Z = \frac{1}{4\pi} \oint 2F^- = p - iq . \quad (3.25)$$

Since the mass of the extreme Reissner-Nordstrom black hole is $M = \sqrt{q^2 + p^2} = |Z|$ we see that the extreme limit coincides with the supersymmetric BPS limit. The extreme Reissner-Nordstrom black hole therefore has all the properties expected for a BPS state: It is invariant under half of the supertransformations, sits in a short multiplet and saturates the supersymmetric mass bound. We therefore expect that it is absolutely stable, as a solution of $N = 2$ supergravity. Since the surface gravity and, hence, the Hawking temperature vanishes it is stable against Hawking radiation. It is very likely, however, that charged black holes in non-supersymmetric gravity are unstable due to charge superradiance. But in a theory with $N \geq 2$ supersymmetry there is no state of lower energy and the black hole is absolutely stable. Note also that the no-force property of multi-center solutions can now be understood as a consequence of the additional supersymmetry present in the system.

Finally we would like to point out that supersymmetry also accounts for the special properties of the near horizon solution. Whereas the BPS black hole has four Killing spinors at generic values of the radius r , this is different at infinity and at the horizon. At infinity the solution approaches flat space, which has 8 Killing spinors. But also the Bertotti-Robinson geometry, which is approached at the horizon, has 8 Killing spinors. Thus the number of unbroken supersymmetries doubles in the asymptotic regions. Since the Bertotti-Robinson solution has the maximal number of Killing spinors, it is a supersymmetry singlet and an alternative vacuum of $N = 2$ supergravity. Thus, the extreme Reissner-Nordstrom black hole interpolates between vacua: this is another typical property of a soliton.

So far we have seen that one can check that a given solution to the equations of motion is supersymmetric, by plugging it into the Killing spinor equation. Very often one can successfully proceed in the opposite way and systematically construct supersymmetric solutions by first looking at the Killing spinor equation and taking it as a condition on the bosonic background. This way one gets first order differential equations for the background which are more easily solved than the equations of motion themselves, which are second order. Let us illustrate this with an example.

Exercise IX : Consider a metric of the form

$$ds^2 = -e^{-2f(\vec{x})} dt^2 + e^{2f(\vec{x})} d\vec{x}^2, \quad (3.26)$$

with an arbitrary function $f(\vec{x})$. In such a background the time component of the Killing spinor equation takes the form

$$\delta\psi_{tA} = -\frac{1}{2}\partial_i f e^{-2f}\gamma^0\gamma^i\epsilon_A + e^{-f}F_{0i}^-\gamma^i\epsilon_{AB}\epsilon^B \stackrel{!}{=} 0. \quad (3.27)$$

In comparison to (3.20) we have chosen the time component and explicitly evaluated the spin connection. The indices $0, i = 1, 2, 3$ are flat indices.

Reduce this equation to one differential equation for the background by making an ansatz for the Killing spinor. Show that the resulting equation together with the Maxwell equation for the graviphoton field strength implies that this solution is precisely the Majumdar Papapetrou solution.

As this exercise illustrates, the problem of constructing supersymmetric solutions has two parts. The first question is what algebraic condition one has to impose on the Killing spinor. This is also the most important step in classifying supersymmetric solitons. In a second step one has to determine the bosonic background by solving differential equations. As illustrated in the above exercise the resulting solutions are very often expressed in terms of harmonic functions. We would now like to discuss the first, algebraic step of the problem. This is related to the so-called Nester construction. In order to appreciate the power of this formalism we digress for a moment from our main line of thought and discuss positivity theorems in gravity.

Killing spinors are useful even outside supersymmetric theories. The reason is that one can use the embedding of a non-supersymmetric theory into a bigger supersymmetric theory as a mere tool to derive results. One famous example is the derivation of the positivity theorem for the ADM mass of asymptotically flat space-times by Witten, which, thanks to the use of spinor techniques is much simpler than the original proof by Schoen and Yau. The Nester construction elaborates on this idea.

In order to prove the positivity theorem one makes certain general assumptions: One considers an asymptotically flat space-time, the equations of motion are required to be satisfied and it is assumed that the behaviour of matter is 'reasonable' in the sense that a suitable condition on the energy momentum

tensor (e.g. the so-called dominant energy condition) is satisfied. The Nester construction then tells how to construct a two-form ω_2 , such that the integral over an asymptotic two-sphere satisfies the inequality

$$\oint \omega_2 = \overline{\epsilon(\infty)}[\gamma^\mu P_\mu + ip + q\gamma_5]\epsilon(\infty) \geq 0. \quad (3.28)$$

Here P_μ is the four-momentum of the space-time (which is defined because we assume asymptotic flatness), q, p are its electric and magnetic charge and $\epsilon(\infty)$ is the asymptotic value of a spinor field used as part of the construction. (The spinor is a Dirac spinor.) The matrix between the spinors is called the Bogomol'nyi matrix (borrowing again terminology from Yang-Mills theory). It has eigenvalues $M \pm \sqrt{q^2 + p^2}$ and therefore we get precisely the mass bound familiar from the Reissner-Nordstrom black hole. But note that this result has been derived based on general assumptions, not on a particular solution. Equality holds if and only if the spinor field $\epsilon(x)$ satisfies the Killing spinor equation (3.20). The static space-times satisfying the bound are precisely the Majumdar-Papapetrou solutions.

The relation to supersymmetry is obvious: we have seen above that the matrix of supersymmetry anticommutators has eigenvalues $M \pm |Z|$, (3.14) and that in supergravity the central charge is $Z = p - iq$, (3.25). Thus the Bogomol'nyi matrix must be related to the matrix of supersymmetry anticommutators.

Exercise X : Express the Bogomol'nyi matrix in terms of supersymmetry anticommutators.

The algebraic problem of finding the possible projections of Killing spinors is equivalent to finding the possible eigenvectors with eigenvalue zero of the Bogomol'nyi matrix. Again we will study one particular example in an exercise.

Exercise XI : Find a zero eigenvector of the Bogomol'nyi matrix which describes a massive BPS state at rest.

In the case of pure $N = 2$ supergravity all supersymmetric solutions are known. Besides the Majumdar-Papapetrou solutions there are two further classes of solutions: The Israel-Wilson-Perjes (IWP) solutions, which are rotating, stationary generalizations of the Majumdar-Papapetrou solutions and the plane fronted gravitational waves with parallel rays (pp-waves).

3.3 Literature

The representation theory of the extended supersymmetry algebra is treated in chapter 2 of Wess and Bagger [4]. The interpretation of the extreme Reissner-Nordstrom black hole as a supersymmetric soliton is due to Gibbons [5]. Then Gibbons and Hull showed that the Majumdar-Papapetrou solutions and pp-waves are supersymmetric [6]. They also discuss the relation to the positivity theorem for the ADM mass and the Nester construction. The classification of supersymmetric solitons in pure $N = 2$ supergravity was completed by Tod [7]. The

Majumdar-Papetrou solutions are discussed in some detail in [2]. Our discussion of Killing spinors uses the conventions of Behrndt, Lüster and Sabra, who have treated the more general case where vector multiplets are coupled to $N = 2$ supergravity [8]. A nice exposition of how supersymmetric solitons are classified in terms of zero eigenvectors of the Bogomol'nyi matrix has been given by Townsend for the case of eleven-dimensional supergravity [9].

4 p -branes in type II string theory

In this section we will consider p -branes, which are higher dimensional cousins of the extremal Reissner-Nordstrom black hole. These p -branes are supersymmetric solutions of ten-dimensional supergravity, which is the low energy limit of string theory. We will restrict ourselves to the string theories with the highest possible amount of supersymmetry, called type IIA and IIB. We start by reviewing the relevant elements of string theory.

4.1 Some elements of string theory

The motion of a string in a curved space-time background with metric $G_{\mu\nu}(X)$ is described by a two-dimensional non-linear sigma-model with action

$$S_{WS} = \frac{1}{4\pi\alpha'} \int_{\Sigma} d^2\sigma \sqrt{-h} h^{\alpha\beta}(\sigma) \partial_{\alpha} X^{\mu} \partial_{\beta} X^{\nu} G_{\mu\nu}(X). \quad (4.1)$$

The coordinates on the world-sheet Σ are $\sigma = (\sigma^0, \sigma^1)$ and $h_{\alpha\beta}(\sigma)$ is the intrinsic world-sheet metric, which locally can be brought to the flat form $\eta_{\alpha\beta}$. The coordinates of the string in space-time are $X^{\mu}(\sigma)$. The parameter α' has the dimension L^2 (length-squared) and is related to the string tension τ_{F1} by $\tau_{F1} = \frac{1}{2\pi\alpha'}$. It is the only independent dimensionful parameter in string theory. Usually one uses string units, where α' is set to a constant (in addition to $c = \hbar = 1$).¹¹ In the case of a flat space-time background, $G_{\mu\nu} = \eta_{\mu\nu}$, the world-sheet action (4.1) reduces to the action of D free two-dimensional scalars and the theory can be quantized exactly. In particular one can identify the quantum states of the string.

At this point one can define different theories by specifying the types of world sheets that one admits. Both orientable and non-orientable world-sheets are possible, but we will only consider orientable ones. Next one has the freedom of adding world-sheet fermions. Though we are interested in type II superstrings, we will for simplicity first consider bosonic strings, where no world-sheet fermions are present. Finally one has to specify the boundary conditions along the space direction of the world sheet. One choice is to impose Neumann boundary conditions,

$$\partial_1 X^{\mu}|_{\partial\Sigma} = 0. \quad (4.2)$$

¹¹We will see later that it is in general not possible to use Planckian and stringy units simultaneously. The reason is that the ratio of the Planck and string scale is the dimensionless string coupling, which is related to the vacuum expectation value of the dilaton and which is a free parameter, at least in perturbation theory.

This corresponds to open strings. In the following we will be mainly interested in the massless modes of the strings, because the scale of massive excitations is naturally of the order of the Planck scale. The massless state of the open bosonic string is a gauge boson A_μ .

Another choice of boundary conditions is Dirichlet boundary conditions,

$$\partial_0 X^\mu|_{\partial\Sigma} = 0 . \quad (4.3)$$

In this case the endpoints of the string are fixed. Since momentum at the end is not conserved, such boundary conditions require to couple the string to another dynamical object, called a D -brane. Therefore Dirichlet boundary conditions do not describe strings in the vacuum but in a solitonic background. Obviously the corresponding soliton is a very exotic object, since we can describe it in a perturbative picture, whereas conventional solitons are invisible in perturbation theory. As we will see later D -branes have a complementary realization as higher-dimensional analogues of extremal black holes. The perturbative D -brane picture of black holes can be used to count microstates and to derive the microscopic entropy.

In order to prepare for this let us consider a situation where one imposes Neumann boundary conditions along time and along p space directions and Dirichlet boundary conditions along the remaining $D - p - 1$ directions (D is the dimension of space-time). More precisely we require that open strings end on the p -dimensional plane $X^m = X_0^m$, $m = p+1, \dots, D-p-1$. This is called a Dirichlet- p -brane or Dp -brane for short. The massless states are obtained from the case of pure Neumann boundary conditions by dimensional reduction: One gets a p -dimensional gauge boson A^μ , $\mu = 0, 1, \dots, p$ and $D - p - 1$ scalars ϕ^m . Geometrically the scalars describe transverse oscillations of the brane.

As a generalization one can consider N parallel Dp -branes. Each brane carries a $U(1)$ gauge theory on its worldvolume, and as long as the branes are well separated these are the only light states. But if the branes are very close, then additional light states result from strings that start and end on different branes. These additional states complete the adjoint representation of $U(N)$ and therefore the light excitations of N near-coincident Dp branes are described by the dimensional reduction of $U(N)$ gauge theory from D to $p + 1$ dimensions.

The final important class of boundary conditions are periodic boundary conditions. They describe closed strings. The massless states are the graviton $G_{\mu\nu}$, an antisymmetric tensor $B_{\mu\nu}$ and a scalar ϕ , called the dilaton. As indicated by the notation a curved background as in the action (4.1) is a coherent states of graviton string states. One can generalize this by adding terms which describe the coupling of the string to other classical background fields. For example the couplings to the B -field and to the open string gauge boson A_μ are

$$S_B = \frac{1}{4\pi\alpha'} \int_\Sigma d^2\sigma \varepsilon^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu B_{\mu\nu}(X) \quad (4.4)$$

and

$$S_A = \oint_{\partial\Sigma} d^1\sigma^\alpha \partial_\alpha X^\mu A_\mu(X) . \quad (4.5)$$

Interactions of strings are encoded in the topology of the world-sheet. The S-matrix can be computed by a path integral over all world sheets connecting given initial and final states. For the low energy sector all the relevant information is contained in the low energy effective action of the massless modes. We will see examples later. The low energy effective action is derived by either matching string theory amplitudes with field theory amplitudes or by imposing that the non-linear sigma model, which describes the coupling of strings to the background fields $G_{\mu\nu}, B_{\mu\nu}, \dots$ is a conformal field theory. Conformal invariance of the world sheet theory is necessary for keeping the world sheet metric $h_{\alpha\beta}$ non-dynamical.¹² A set of background fields $G_{\mu\nu}, B_{\mu\nu}, \dots$ which leads to an exact conformal field theory provides an exact solution to the classical equations of motion of string theory. Very often one only knows solutions of the low energy effective field theory.

Exercise XII : Consider a curved string background which is independent of the coordinate X^1 , and with $G_{1\nu} = 0$, $B_{1\nu} = 0$ and $\phi = \text{const}$. Then the G_{11} -part of the world-sheet action factorizes,¹³

$$S[G_{11}] = \int d^2\sigma G_{11}(X^m) \partial_+ X^1 \partial_- X^1, \quad (4.6)$$

where $m \neq 1$. We have introduced light-cone coordinates σ^\pm on the world-sheet. The isometry of the target space $X^1 \rightarrow X^1 + a$, where a is a constant, is a global symmetry from the world-sheet point of view. Promote this to a local shift symmetry, $X^1 \rightarrow X^1 + a(\sigma)$ ('gauging of the global symmetry') by introducing suitable covariant derivatives D_\pm . Show that the locally invariant action

$$\hat{S} = \int d^2\sigma \left(G_{11} D_+ X^1 D_- X^1 + \tilde{X}^1 F_{+-} \right), \quad (4.7)$$

where $F_{+-} = [D_+, D_-]$ reduces to the globally invariant action (4.6), when eliminating the Lagrange multiplier \tilde{X}^1 through its equation of motion. Next, eliminate the gauge field A_\pm from (4.7) through its equation of motion. What is the interpretation of the resulting action?

The above exercise illustrates T-duality in the most simple example. T-duality is a stringy symmetry, which identifies different values of the background fields $G_{\mu\nu}, B_{\mu\nu}, \phi$ in a non-trivial way. The version of T-duality that we consider here applies if the space-time background has an isometry or an abelian group of isometries. This means that when using adapted coordinates the metric and all other background fields are independent of one or of several of the embedding coordinates X^μ . For later reference we note the transformation law of the fields under a T-duality transformation along the 1-direction:

$$G'_{11} = \frac{1}{G_{11}}, \quad G'_{1m} = \frac{B_{1m}}{G_{11}}, \quad B'_{1m} = \frac{G_{1m}}{G_{11}},$$

¹²It might be possible to relax this and to consider the so-called non-critical or Liouville string theory. But then one gets a different and much more complicated theory.

¹³We have set α' to a constant for convenience.

$$G'_{mn} = G_{mn} - \frac{G_{m1}G_{1n} + B_{m1}B_{1n}}{G_{11}}, \quad B'_{mn} = B_{mn} - \frac{G_{m1}B_{1n} + B_{m1}G_{1n}}{G_{11}},$$

$$\phi' = \phi - \log \sqrt{G_{11}}, \quad (4.8)$$

where $m \neq 1$. These formulae apply to closed bosonic strings. In the case of open strings T-duality mutually exchanges Neumann and Dirichlet boundary conditions. This is one of the motivations for introducing D -branes.

Let us now discuss the extension from the bosonic to the type II string theory. In type II theory the world sheet action is extended to a $(1,1)$ supersymmetric action by adding world-sheet fermions $\psi^\mu(\sigma)$. It is a theory of closed oriented strings. For the fermions one can choose the boundary conditions for the left-moving and right-moving part independently to be either periodic (Ramond) or antiperiodic (Neveu-Schwarz). This gives four types of boundary conditions, which are referred to as NS-NS, NS-R, R-NS and R-R in the following. Since the ground state of an R-sector carries a representation of the D -dimensional Clifford algebra, it is a space-time spinor. Therefore the states in the NS-NS and R-R sector are bosonic, whereas the states in the NS-R and R-NS sector are fermionic.

Unitarity of the quantum theory imposes consistency conditions on the theory. First the space-time dimension is fixed to be $D = 10$. Second one has to include all possible choices of the boundary conditions for the world-sheet fermions. Moreover the relative weights of the various sectors in the string path integral are not arbitrary. Among the possible choices two lead to supersymmetric theories, known as type IIA and type IIB. Both differ in the relative chiralities of the R-groundstates: The IIB theory is chiral, the IIA theory is not.

The massless spectra of the two theories are as follows: The NS-NS sector is identical for IIA and IIB:

$$\text{NS-NS} : G_{\mu\nu}, B_{\mu\nu}, \phi. \quad (4.9)$$

The R-R sector contains various n -form gauge fields

$$A_n = \frac{1}{n!} A_{\mu_1 \dots \mu_n} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_n} \quad (4.10)$$

and is different for the two theories:

$$\text{R-R} : \begin{cases} \text{IIA} : & A_1, A_3 \\ \text{IIB} : & A_0, A_2, A_4 \end{cases} \quad (4.11)$$

The 0-form A_0 is a scalar with a Peccei-Quinn symmetry, i.e. it enters the action only via its derivative. The 4-form is constrained, because the corresponding field strength F_5 is required to be selfdual: $F_5 = \star F_5$. Finally, the fermionic sectors contain two gravitini and two fermions, called dilatini:

$$\text{NS-R/R-NS} : \begin{cases} \text{IIA} : & \psi_+^{(1)\mu}, \psi_-^{(2)\mu}, \psi_+^{(1)}, \psi_-^{(2)}, \\ \text{IIB} : & \psi_+^{(1)\mu}, \psi_+^{(2)\mu}, \psi_+^{(1)}, \psi_+^{(2)}. \end{cases} \quad (4.12)$$

More recently it has been proposed to add D -branes and the corresponding open string sectors to the type II theory. The motivation for this is the existence of p -brane solitons in the type II low energy effective theory. In the next sections we will study these p -branes in detail and review the arguments that relate them to Dp -branes. One can show that the presence of a Dp -brane or of several parallel Dp -branes breaks only half of the ten-dimensional supersymmetry of type IIA/B theory, if one chooses p to be even/odd, respectively. Therefore such backgrounds describe BPS states. The massless states associated with a Dp -brane correspond to the dimensional reduction of a ten-dimensional vector multiplet from ten to $p + 1$ dimensions. In the case of N near-coincident Dp -branes one gets the dimensional reduction of a supersymmetric ten-dimensional $U(N)$ gauge theory.

T-duality can be extended to type II string theories. There is one important difference to the bosonic string: T-duality is not a symmetry of the IIA/B theory, but maps IIA to IIB and vice versa.

This concludes our mini-introduction to string theory. From now on we will mainly consider the low energy effective action.

4.2 The low energy effective action

The low energy effective action of type IIA/B superstring theory is type IIA/B supergravity. The p -branes which we will discuss later in this section are solitonic solutions of supergravity. We need to make some introductory remarks on the supergravity actions. Since we are interested in bosonic solutions, we will only discuss the bosonic part. We start with the NS-NS sector, which is the same for type IIA and type IIB and contains the graviton $G_{\mu\nu}$, the antisymmetric tensor $B_{\mu\nu}$ and the dilaton ϕ . One way to parametrize the action is to use the so-called string frame:

$$S_{NS-NS} = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-G} e^{-2\phi} \left(R + 4\partial_\mu \phi \partial^\mu \phi - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} \right). \quad (4.13)$$

The three-form $H = dB$ is the field strength of the B -field. The metric $G_{\mu\nu}$ is the string frame metric, that is the metric appearing in the non-linear sigma-model (4.1), which describes the motion of a string in a curved background. The string frame action is adapted to string perturbation theory, because it depends on the dilaton in a uniform way. The vacuum expectation value of the dilaton defines the dimensionless string coupling,

$$g_S = e^{\langle \phi \rangle}. \quad (4.14)$$

The terms displayed in (4.13) are of order g_S^{-2} and arise at string tree level. Higher order g -loop contributions are of order g_S^{-2+2g} and can be computed using string perturbation theory. The constant κ_{10} has the dimension of a ten-dimensional gravitational coupling. Note, however, that it can not be directly identified with the physical gravitational coupling, because a rescaling of κ_{10} can be compensated by a shift of the dilaton's vacuum expectation value $\langle \phi \rangle$.

This persists to higher orders in string perturbation theory, because ϕ and κ_{10} only appear in the combination $\kappa_{10}e^\phi$. One can use this to eliminate the scale set by the dimensionful coupling in terms of the string scale $\sqrt{\alpha'}$ by imposing

$$\kappa_{10} = (\alpha')^2 g_S \cdot \text{const} . \quad (4.15)$$

There is only one independent dimensionful parameter and only one single theory, which has a family of ground states parametrized by the string coupling.

It should be noted that (4.13) has not the canonical form of a gravitational action. In particular the first term is not the standard Einstein-Hilbert term. This is the second reason why the constant κ_{10} in front of the action is not necessarily the physical gravitational coupling. Moreover the definitions of mass and Bekenstein-Hawking entropy are tied to a canonically normalized gravitational action. Therefore we need to know how to bring (4.13) to canonical form by an appropriate field redefinition. For later use we discuss this for general space-time dimension D . Given the D -dimensional version of (4.13) the canonical or Einstein metric is

$$g_{\mu\nu} = G_{\mu\nu} e^{-4(\phi - \langle \phi \rangle)/(D-2)} \quad (4.16)$$

and the Einstein frame action is

$$S_{NS-NS} = \frac{1}{2\kappa_{D,phys}^2} \int d^D x \sqrt{-g} (R(g) + \dots) . \quad (4.17)$$

We have absorbed the dilaton vacuum expectation value in the prefactor of the action to get the physical gravitational coupling.¹⁴ The action now has canonical form, but the uniform dependence on the string coupling is lost.

Let us now turn to the R-R sector, which consists of n -form gauge fields A_n , with $n = 1, 3$ for type A and $n = 0, 2, 4$ for type B. The standard kinetic term for an n -form gauge field in D dimensions is

$$S \simeq \int_D F_{n+1} \wedge \star F_{n+1} , \quad (4.18)$$

where the integral is over D -dimensional space and $F_{n+1} = dA_n$. The R-R action in type II theories contains further terms, in particular Chern-Simons terms. Moreover the gauge transformations are more complicated than $A_n \rightarrow A_n + df_{n-1}$, because some of the A_n are not inert under the transformations of the others. For simplicity we will ignore these complications here and only discuss simple n -form actions of the type (4.18).¹⁵ We need, however, to make two further remarks. The first is that the action (4.18) is neither in the string nor in the Einstein frame. Though the Hodge- \star is build using the string metric, there is no explicit dilaton factor in front. The reason is that if one makes the

¹⁴There is a second, slightly different definition of the Einstein frame where the dilaton expectation values is absorbed in $g_{\mu\nu}$ and not in the gravitational coupling. This second definition is convenient in the context of IIB S-duality, because the resulting metric $g_{\mu\nu}$ is invariant under S-duality. The version we use in the text is the correct one if one wants to use the standard formulae of general relativity to compute mass and entropy.

¹⁵The full R-R action discussion is discussed in [10].

dilaton explicit, then the gauge transformation law involves the dilaton. It is convenient to have the standard gauge transformation and as a consequence the standard form of the conserved charge. Therefore the dilaton has been absorbed in the gauge field A_n in (4.18), although this obscures the fact that the term arises at string tree level. The second remark concerns the four-form A_4 in type IIB theory. Since the associated field strength F_5 is selfdual, $F_5 = *F_5$, it is non-trivial to write down a covariant action. The most simple way to proceed is to use a term $S \simeq \frac{1}{2} \int_{10} F_5 \wedge *F_5$ in the action and to impose $F_5 = *F_5$ at the level of the field equations.¹⁶

Let us now discuss what are the analogues of point-like sources for an action of the type (4.18). In general electric sources which couple minimally to the gauge field A_n are described by a term

$$\int_D A_n \wedge *j_n , \quad (4.19)$$

where the electric current j_n is an n -form. Variation of the combined action (4.18),(4.19) yields Euler Lagrange equations with a source term,

$$d *F_{n+1} = *j_n . \quad (4.20)$$

The Bianchi identity is $dF_{n+1} = 0$. Analogues of point sources are found by localizing the current on a $(p = n - 1)$ -dimensional spacelike surface with $(p + 1 = n)$ -dimensional world volume:

$$\int_D A_{p+1} \wedge *j_{p+1} = \int_{p+1} A_{p+1} . \quad (4.21)$$

Thus sources are p -dimensional membranes, or p -branes for short. We consider the most simple example where space-time is flat and the source is the p -dimensional plane $x^i = 0$ for $i = p + 1, \dots, D - 1$. It is convenient to introduce spheric coordinates in the directions transverse to the brane, $x^i = (r, \phi^1, \dots, \phi^{D-p-2})$. Then the generalized Maxwell equations reduce to

$$\Delta^\perp A_{01\dots p}(r) \simeq \delta(r) , \quad (4.22)$$

where Δ^\perp is the Laplace Operator with respect to the transverse coordinates and the indices $0, 1, \dots, p$ belong to directions parallel to the world volume. In the following we will not keep track of the precise factors in the equations. This is indicated by the symbol \simeq . The gauge field and field strength solving (4.22) are

$$A_{01\dots p} \simeq \frac{Q}{r^{D-p-3}} \text{ and } F_{0r1\dots p} \simeq \frac{Q}{r^{D-p-2}} . \quad (4.23)$$

More generally one might consider a curved space-time or sources which have a finite extension along the transverse directions. If the solution has isometry

¹⁶In [11] a proposal has been made how to construct covariant actions for this type of theories.

group $\mathbb{R}_t \times ISO(p) \times SO(D-p-1)$ and approaches flat space in the transverse directions, then its asymptotic form is given by (4.23).

The parameter Q is the electric charge (or more precisely the electric charge density). As in electrodynamics one can write the charge as a surface integral,

$$Q \simeq \oint_{D-p-2} {}^*F_{p+2} , \quad (4.24)$$

where the integration is over a $(D-p-2)$ -surface which encloses the source in transverse space. We take this integral as our definition of p -brane charge.

Magnetic sources are found by exchanging the roles of equation of motion and Bianchi identity. They couple minimally to the magnetic potential \tilde{A} , where $d\tilde{A} = {}^*F$. Localized sources are \tilde{p} -branes with $\tilde{p} = D-p-4$. The potential and field strength corresponding to a flat \tilde{p} -brane in flat space-time are

$$\tilde{A}_{01\dots\tilde{p}} \simeq \frac{P}{r^{p+1}} \text{ and } F_{\phi^1\dots\phi^{p+2}} \simeq {}^*F_{0r1\dots\tilde{p}} \simeq \frac{P}{r^{p+2}} . \quad (4.25)$$

The magnetic charge is

$$P \simeq \oint_{p+2} F_{p+2} . \quad (4.26)$$

Generically, electric and magnetic sources have different dimensions, $p \neq \tilde{p}$. Dyonic objects are only possible for special values D and p , for example 0-branes (particles) in $D = 4$ and 3-branes in $D = 10$. Electric and magnetic charges are restricted by a generalized Dirac quantization condition,

$$PQ \simeq n , \text{ with } n \in \mathbb{Z} . \quad (4.27)$$

This can be derived by either generalizing the Dirac string construction or the Wu-Yang construction.

We now turn to the discussion of p -brane solitons in type II supergravity. By p -brane we indicate that we require that the soliton has isometries $\mathbb{R}_t \times ISO(p) \times SO(D-p-1)$. As before we take a soliton to be a solution to the equations of motion, which has finite energy per worldvolume, has no naked singularities and is stable. As in the Reissner-Nordstrom case one can find solutions which have Killing spinors and therefore are stable as a consequence of the BPS bound.¹⁷ The p -branes are charged with respect to the various tensor fields appearing in the type IIA/B action. Since we know which tensor fields exist in the IIA/B theory, we know in advance which solutions we have to expect. The electric and magnetic source for the B -field are a 1-brane or string, called the fundamental string and a 5-brane, called the solitonic 5-brane or NS-5-brane. In the R-R-sector there are R-R-charged p -branes with $p = 0, \dots, 6$ with p even/odd for type IIA/B. Before discussing them we comment on some exotic objects, which we won't discuss further. First there are R-R-charged (-1) -branes and 7-branes, which are electric and magnetic sources for the type IIB R-R-scalar A_0 . The (-1) -brane is localized in space and time and therefore it is interpreted, after

¹⁷For these extremal p -branes the isometry group is enhanced to $ISO(1, p) \times SO(D-p-1)$.

going to Euclidean time, as an instanton. The 7-brane is also special, because it is not asymptotically flat. This is a typical feature of brane solutions with less than 3 transverse directions, for example black holes in $D = 3$ and cosmic strings in $D = 4$. Both the (-1) -brane and the 7-brane are important in string theory. First it is believed that R-R-charged p -branes describe the same BPS-states as the Dp -branes defined in string perturbation theory. Therefore one needs supergravity p -branes for all values of p . Second the (-1) -brane can be used to define and compute space-time instanton corrections in string theory, whereas the 7-brane is used in the F-theory construction of non-trivial vacua of the type IIB string. One also expects to find R-R-charged p -branes with $p = 8, 9$. For those values of p there are no corresponding gauge fields. The gauge field strength F_{10} related to an 8-brane has been identified with the cosmological constant in the massive version of IIA supergravity. Finally the 9-brane is just flat space-time.

4.3 The fundamental string

The fundamental string solution is electrically charged with respect to the NS-NS B -field. Its string frame metric is

$$ds_{Str}^2 = H_1^{-1}(x_i)(-dt^2 + dy^2) + \sum_{i=1}^8 dx_i^2, \quad (4.28)$$

where H_1 is a harmonic function with respect to the eight transverse coordinates,

$$\Delta_{x_i}^\perp H_1 = 0. \quad (4.29)$$

Single center solutions are described by the spherically symmetric choice

$$H_1(r) = 1 + \frac{Q_1}{r^6}, \quad (4.30)$$

where Q_1 is a positive constant. To fully specify the solution we have to display the B -field and the dilaton:

$$B_{ty} = H_1^{-1} - 1 \text{ and } e^{-2\phi} = H_1. \quad (4.31)$$

All other fields are trivial. Since only NS-NS fields are excited, this is a solution of both IIA and IIB theory. In order to interpret the solution we have to compute its tension and its charge. Both quantities can be extracted from the behaviour of the solution at infinity.

The analogue of mass for a p -brane is the mass per world volume, or tension T_p . Generalizing our discussion of four-dimensional asymptotically flat space-times, the tension can be extracted by compactifying the p world volume directions and computing the mass of the resulting pointlike object in $d = D - p$ dimensions, using the d -dimensional version of (2.9),

$$g_{00} = -1 + \frac{16\pi G_N^{(d)}}{(d-2)\omega_{d-2}} \frac{M}{r^{d-3}} + \dots. \quad (4.32)$$

Note that this formula refers to the Einstein metric. As explained above the standard definitions of mass and energy are tied to the Einstein frame metric. $G_N^{(d)}$ is the d -dimensional Newton constant and ω_{d-2} is the volume of the unit sphere $S^{d-2} \subset \mathbb{R}^{d-1}$,

$$\omega_n = \frac{2\pi^{(n+1)/2}}{\Gamma(\frac{n+1}{2})} . \quad (4.33)$$

The quantity r_S , where

$$r_S^{d-3} = \frac{16\pi G_N^{(d)} M}{(d-2)\omega_{d-2}} \quad (4.34)$$

has dimension length and is the d -dimensional Schwarzschild radius.

The tension of the p -brane in D dimensions and the mass of the compactified brane in d dimensions are related by $G_N^{(D)} T_p = G_N^{(d)} M$, because $G_N^{(D)} = V_p G_N^{(d)}$ and $T_p = M/V_p$, where V_p is the volume of the internal space. Dimensional reduction of actions and branes will be discussed in some detail in the next section.

The p -brane charge is computed by measuring the flux of the corresponding field strength through an asymptotic sphere in the transverse directions. Since in the limit $r \rightarrow \infty$ the B -field takes the form (4.23), it is natural to interpret Q_1 as the electric B -field charge. This is the terminology that we will adopt, but we need to make two clarifying remarks. First note that the parameter Q_1 in the harmonic function is always positive. Solutions of negative charge are described by flipping the sign of B_{ty} , which is not fixed by the equations of motion. If one wants to denote the negative charge by $Q_1 < 0$, then one needs to replace in the harmonic function Q_1 by $-Q_1$. Obviously it is convenient and no loss in generality to restrict oneself to the case $Q_1 > 0$. The second remark is that in the string frame NS-NS action the kinetic term of the B -field is dressed with a dilaton dependent factor $e^{-2\phi}$. Therefore one must include this factor in order to get a conserved charge. But since the parameter Q_1 is proportional to the conserved charge,

$$Q_1 \simeq \oint_7 \star(e^{-2\phi} H) , \quad (4.35)$$

we can take Q_1 to be the B -field charge by appropriate choice of the normalization constant.

A similar convention can be used for general p -brane solutions. As we will see later, p -brane solutions in D dimensions are characterized by harmonic functions

$$H = 1 + \frac{Q_p}{r^{D-p-3}} . \quad (4.36)$$

We will always take $Q_p > 0$ and refer to it as the D -dimensional p -brane charge. With this convention the charge has dimension L^{D-p-3} .

Let us next study the behaviour of the fundamental string solution for $r \rightarrow 0$. It turns out that there is a so-called null singularity, a curvature singularity which coincides with an event horizon and therefore is not a naked singularity.

The fundamental string is the extremality limit of a family of so-called black string solutions, which satisfy the inequality

$$T_1 \geq \mathcal{C}Q_1 \quad (4.37)$$

between tension and charge, where \mathcal{C} is a constant. Black strings with $T_1 > \mathcal{C}Q_1$ have an outer event horizon and an inner horizon which coincides with a curvature singularity. In the extreme limit $T_1 = \mathcal{C}Q_1$ the two horizons and the singularity coincide and one obtains the fundamental string. As for the Reissner-Nordstrom black hole the extreme limit is supersymmetric. Type IIA/B supergravity has 32 real supercharges and the fundamental string is a 1/2 BPS solution with 16 Killing spinors. In the IIA theory, the explicit form of the Killing spinors is

$$\epsilon_{\pm} = H_1^{-1/4}(r)\epsilon_{\pm}(\infty) , \quad (4.38)$$

with

$$\Gamma^0\Gamma^9\epsilon_{\pm}(\infty) = \pm\epsilon_{\pm}(\infty) . \quad (4.39)$$

The spinors ϵ_{\pm} are ten-dimensional Majorana-Weyl spinors of opposite chirality. The 9-direction is the direction along the worldvolume, $x_9 = y$.

One can also find static multi-center solutions which generalize the Majumdar-Papapetrou solution. The corresponding harmonic functions are

$$H = 1 + \sum_{I=1}^N \frac{Q_1^I}{|\vec{x} - \vec{x}_I|^6} , \quad (4.40)$$

where \vec{x}, \vec{x}_I are eight-dimensional vectors. The positions \vec{x}_I of the strings in the eight-dimensional transverse space are completely arbitrary.

Moreover it is possible to interpret Q_1 within the supersymmetry algebra. The IIA supersymmetry algebra can be extended by charges Z_{μ} , which transform as Lorentz vectors. For a fundamental string along the y -direction the charge (actually: charge density) is

$$Z_{\mu} \simeq (0, \dots, 0, Q_1) . \quad (4.41)$$

Such charges are often called central charges, though they are not literally central, because they do not commute with Lorentz transformations. The extended IIA algebra takes the form

$$\begin{aligned} \{Q_{\alpha}^{+}, Q_{\beta}^{+}\} &= (P + Z)_{\mu} (CP^{+}\Gamma^{\mu})_{\alpha\beta} \\ \{Q_{\alpha}^{-}, Q_{\beta}^{-}\} &= (P - Z)_{\mu} (CP^{-}\Gamma^{\mu})_{\alpha\beta} , \end{aligned} \quad (4.42)$$

where C is the charge conjugation matrix and P^{\pm} projects onto positive/negative chirality. The p -brane charges Z_{μ} are not excluded by the classification theorem of Haag, Lopuzanski and Sohnius, because they are carried by field configurations which do not approach the vacuum in all directions, but

only in the transverse directions. If one compactifies along the worldvolume directions they become central charges in the usual sense of the lower dimensional supersymmetry algebra.

So far we have analysed the fundamental string within the framework of supergravity. We now turn to its interpretation within string theory. Although we call the fundamental string a soliton (in the broad sense explained above) it is not a regular solution of the field equations, but singular at $r = 0$. The singularity can be interpreted in terms of a source concentrated at the origin. This source term is nothing but the type IIA/B world sheet action itself.

We have seen how this works in the simplified case without gravity: The integral of the gauge field over the world-sheet (compare (4.21) to (4.4))

$$\frac{1}{2\pi\alpha'} \int_2 B = \frac{1}{4\pi\alpha'} \int_{WS} d^2\sigma B_{\mu\nu}(X) \partial_\alpha X^\mu \partial_\beta X^\nu \varepsilon^{\alpha\beta} \quad (4.43)$$

yields upon variation a δ -function source in the generalized Maxwell equations, see (4.20),(4.22). Similarly the full world sheet action is the appropriate source for the fundamental string solution. Therefore the fundamental string solution of the effective supergravity theory is interpreted as describing the long range fields outside a fundamental type IIA/B string, as already indicated by its name. As a consequence the tension T_1 of the supergravity string solution must be an integer multiple $T_1 = \hat{Q}_1 \tau_{F1}$ of the IIA/B string tension $\tau_{F1} = \frac{1}{2\pi\alpha'}$, where the integer \hat{Q}_1 counts the number of fundamental IIA/B strings placed at $r = 0$. From formula (4.43) it is obvious that τ_{F1} measures the coupling of the B -field to the string world sheet and therefore it can be interpreted as the fundamental electric charge unit. This provides a somewhat different definition of the charge than the one by the parameter Q_1 . Since both kinds of definition are used in the literature, we will now explain how they are related for a generic p -brane.

Consider a p -brane which is charged with respect to a $(p+1)$ -form gauge potential A_{p+1} . The coupling between the brane and the gauge field is described by

$$\tilde{Q}_p \int_{p+1} A_{p+1} , \quad (4.44)$$

where \tilde{Q}_p has the dimension L^{-p-1} of mass per worldvolume, or tension and measures the strength of the source. Like the parameter Q_p also \tilde{Q}_p measures the conserved charge associated with the gauge field. But both quantities have a different dimension and therefore differ by appropriate powers of α' . By dimensional analysis the relation between the (transverse) Schwarzschild radius r_S , tension T_p and the charges Q_p, \tilde{Q}_p is

$$r_S^{D-p-3} \simeq Q_p \simeq G_N^{(D)} T_p \simeq G_N^{(D)} \tilde{Q}_p , \quad (4.45)$$

up to dimensionless quantities. The D -dimensional Newton constant is related to the D -dimensional gravitational coupling κ_D by $\kappa_D^2 = 8\pi G_N^{(D)}$. We will see how κ_D is related to κ_{10} in the next section, which discusses dimensional reduction. The relation between κ_{10} and α' was given in (4.15). The dimensionless

quantities not specified in (4.45) fall into two classes: First there are numerical factors, which depend in part on conventional choices like, for example, the choice of the constant in (4.15). They are only important if the precise numerical values of tensions or charges are relevant. We will see two examples: the comparison of p -branes and D -branes and the entropy of five-dimensional black holes. We will not keep track of these factors ourselves, but quote results from the literature when needed. The second kind of dimensionless quantity which we suppressed in (4.45) is the dimensionless string coupling g_S . As we will see this dependence is very important for the qualitative behaviour and physical interpretation of a p -brane.

Let us return to the specific case of the fundamental string solution, which carries tension and charge $T_1 = \hat{Q}_1 = \hat{Q}_1 \tau_{F1}$. The dimensionless ratio of tension and charge is independent of the string coupling,

$$T_1 = \tilde{Q}_1 . \quad (4.46)$$

This is specific for fundamental strings. For a soliton (in the narrow sense) one expects that the mass / tension is proportional to g_S^{-2} , whereas (4.46) is the typical behaviour of the fundamental objects of a theory.

A further check of the interpretation of the fundamental string solution is provided by looking at so-called oscillating string solutions, which are obtained by superimposing a gravitational wave on the fundamental string. These solutions have 8 Killing spinors and preserve 1/4 of the supersymmetries of the vacuum. Similarly the perturbative IIA/B string has excitations which sit in 1/4 BPS representations. These are the states which have either only left-moving oscillations or only right-moving oscillations. The spectrum of such excitations matches precisely with the oscillating string solutions.

Finally we mention that the fundamental string solution is not only a solution of supergravity but of the full IIA/B string theory. There is a class of exact two-dimensional conformal field theories, called chiral null models, which includes both the fundamental string and the oscillating strings. This is different for the other supergravity p -branes, where usually no corresponding exact conformal field theory is known.

Exercise XIII : Apply T-duality, both parallel and orthogonal to the world volume, to the fundamental string. Use the formulae (4.8). Why can one T-dualize with respect to a direction orthogonal to the world volume, although this is not an isometry direction?

4.4 The solitonic five-brane

The solitonic five-brane (also called NS-five-brane) is magnetically charged with respect to the NS-NS B -field. Again the solution is parametrized by a harmonic function,

$$ds_{Str}^2 = -dt^2 + \sum_{m=1}^5 dy_m^2 + H_5(x_i) \sum_{i=1}^4 dx_i^2 , \quad (4.47)$$

$$e^{-2\phi} = H_5^{-1} , \quad H_{ijk} = \frac{1}{2} \varepsilon_{ijkl} \partial_l H_5 , \quad (4.48)$$

where

$$\Delta_{x_i}^\perp H_5 = 0 . \quad (4.49)$$

For a single center solution the harmonic function is

$$H_5 = 1 + \frac{Q_5}{r^2} , \quad (4.50)$$

where $Q_5 > 0$ is the magnetic B -charge. Like the fundamental string the solitonic five-brane saturates an extremality bound. And again there are 16 Killing spinors and static multi-center solutions. The condition imposed on the Killing spinor of the IIA theory is

$$\epsilon_\pm = \mp \Gamma^1 \Gamma^2 \Gamma^3 \Gamma^4 \epsilon_\pm , \quad (4.51)$$

where Γ^i correspond to the transverse directions x_i .

But this time there is no need to introduce a source term at $r = 0$, at least when probing this space-times with strings: the solution is geodesically complete in the string frame and a string sigma-model with this target space is well defined. Therefore the five-brane is interpreted as a soliton in the narrow sense of the word, as a fully regular extended solution of the equations of motion, like, for instance, 't Hooft-Polyakov monopoles in Yang-Mills theories.

The electric and magnetic B -field charge are subject to the generalized Dirac quantization condition:

$$Q_1 Q_5 \simeq n . \quad (4.52)$$

Therefore Q_1, Q_5 can only take discrete values. In the last section we arrived at the same conclusion for Q_1 by a different reasoning, namely by identifying the source of the fundamental string solution as the perturbative type II string.

One can introduce fundamental charge units c_1, c_5 , which satisfy (4.52) with $n = 1$. Then the charges carried by fundamental strings and solitonic five-branes are integer multiples of these charge units,

$$Q_i = \hat{Q}_i c_i, \quad \hat{Q}_i \in \mathbb{Z}, \quad i = 1, 5. \quad (4.53)$$

The charge unit c_1 is known from the identification of the fundamental string solution with the perturbative IIA/B string and c_5 is fixed by the quantization law. The fundamental charge units are:¹⁸

$$c_1 = \frac{8G_N^{(10)}}{6\alpha'\omega_7} \text{ and } c_5 = \alpha' , \quad (4.54)$$

where $G_N^{(10)}$ is the ten-dimensional Newton constant, which is related to α' and to the string coupling by

$$G_N^{(10)} = 8\pi^6 g_s^2 (\alpha')^4 . \quad (4.55)$$

¹⁸Here and in the following formula we use the conventions of [12].

One can also define a five-brane charge \tilde{Q}_5 which measures the coupling of the five-brane worldvolume theory to the dual gauge field \tilde{B}_6 , where $d\tilde{B}_6 = \star H$. When considering the \star -dualized version of the NS-NS action, where \tilde{B}_6 is taken to be a fundamental field instead of B_2 , then the solution is not geodesically complete and requires the introduction of a source term proportional to $\int_6 \tilde{B}_6$. As described in the last section one can define a charge \tilde{Q}_5 , which is related to \tilde{Q}_1 by Dirac quantization. \tilde{Q}_1 and \tilde{Q}_5 are integer multiples of a charge units μ_{F1} and μ_{NS5} , which measure the electric and magnetic couplings of B_2 to its sources: $\tilde{Q}_i = \tilde{Q}_i \mu_i$, where $\tilde{Q}_i \in \mathbb{Z}$. The explicit values of the couplings are¹⁹:

$$\mu_{F1} = \frac{1}{2\pi\alpha'} \text{ and } \mu_{NS5} = \frac{1}{(2\pi)^5(\alpha')^3} . \quad (4.56)$$

The relation between five-brane tension T_5 , five-brane charge \tilde{Q}_5 and the string coupling g_S is

$$T_5 = \frac{\tilde{Q}_5}{g_S^2} , \quad (4.57)$$

which is the typical behaviour of the mass or tension of a soliton (in the narrow sense). This is consistent with our interpretation: Fundamental strings, which are electrically charged under the B -field are the fundamental objects of the theory. Therefore five-branes, which are magnetically charged, should be solitons.

Finally we mention that there is an exact conformal field theory which describes the near horizon limit of the five-brane. It consists of a linear dilaton theory corresponding to the transverse radius, an $SU(2)$ Wess-Zumino-Witten model corresponding to the transversal three-sphere and free scalars corresponding to the world volume directions.

4.5 R-R-charged p -branes

We now turn to the class of p -branes which carry R-R-charge, restricting ourselves to the branes with $0 \leq p \leq 6$. Again these solutions saturate an extremality bound, have 16 Killing spinors and admit static multi-center configurations.

The string frame metric is

$$ds_{Str}^2 = H_p^{-1/2}(x_i) \left(-dt^2 + \sum_{m=1}^p dy_m^2 \right) + H_p^{1/2}(x_i) \left(\sum_{i=1}^{D-p-1} dx_i^2 \right) , \quad (4.58)$$

where H_p is harmonic, $\Delta_{x_i}^\perp H_p = 0$. The dilaton and R-R gauge fields are

$$e^{-2\phi} = H_p^{(p-3)/2} \text{ and } A_{01\dots p} = -\frac{1}{2}(H_p^{-1} - 1) . \quad (4.59)$$

When taking the $(p+1)$ -form potentials with $p = 0, 1, 2$ to be fundamental fields, then p -branes are electric for $p = 0, 1, 2$ and magnetic for $p = 4, 5, 6$.

¹⁹These quantities correspond to τ_{F1} and $g^2\tau_{NS5}$ in [10].

For $p = 3$ we have to take into account that the field strength F_5 is self-dual. In this case A_4 is not a gauge potential for F_5 . Instead dA_4 gives the electric components of F_5 and the magnetic components are then fixed by self-duality. Note that the $D3$ -brane is not only dyonic (carrying both electric and magnetic charge) but self-dual, i.e. electric and magnetic charge are dependent.

The electric charges Q_p and the magnetic charges $Q_{\tilde{p}}$ ($\tilde{p} = D - p - 4$) are related by the Dirac quantization condition:

$$Q_p Q_{\tilde{p}} \simeq n, \text{ where } n \in \mathbb{Z}. \quad (4.60)$$

Therefore the spectrum of such charges is discrete and one can introduce fundamental charge units c_p :

$$Q_p = \hat{Q}_p c_p \text{ with } \hat{Q}_p \in \mathbb{Z}. \quad (4.61)$$

Moreover within string theory R-R charged p -brane solutions are interrelated through T-duality and related to the fundamental string and to the NS-five-brane through S-duality.²⁰ Therefore the fundamental charge units are fixed and known. The explicit values are:²¹

$$c_p = g_S (\alpha')^{(7-p)/2} \frac{(2\pi)^{6-p}}{\omega_{6-p}}. \quad (4.62)$$

The charges \tilde{Q}_p , which measure the coupling of A_{p+1} to a p -brane source are integer multiples of the fundamental coupling μ_{Dp} , where

$$\mu_{Dp} = \frac{1}{(2\pi)^p (\alpha')^{(1+p)/2}}. \quad (4.63)$$

The dependence of the tension on the charge $\tilde{Q}_p = \hat{Q}_p \mu_{Dp}$, where $\hat{Q}_p \in \mathbb{Z}$, is

$$T_p = \frac{\tilde{Q}_p}{g_S}. \quad (4.64)$$

This behaviour is between the one of perturbative string states and standard solitons. At this point it is important that in string theories with both open and closed strings the closed string coupling g_S and the open string coupling g_O are related as a consequence of unitarity,

$$g_S = g_O^2 \cdot \text{const}. \quad (4.65)$$

Thus one should try to interpret R-R charged p -branes as solitons²² related to an open string sector of type II theory. The natural way to have open strings in type II theory is to introduce D -branes. This leads to the idea that Dp -branes and R-R p -branes describe the same BPS states.

²⁰The action of T-duality on p -brane solutions is discussed in some of the exercises. S-duality transformations work in a similar way.

²¹Here and in the following we use the conventions of [10].

²²Since for $p \neq 3$ the solution is singular on the horizon, these are not solitons in the narrow sense. As in the case of the fundamental string one needs to introduce a source. This leads to the question whether these branes have the same fundamental status in the theory as strings. It is possible that M-theory is not just a theory of strings.

4.6 Dp -branes and R-R charged p -branes

The idea formulated at the end of the last paragraph can be tested quantitatively by comparing the static R-R force at large distance between two Dp -branes to the one between two R-R charged p -branes. In both cases the full force is exactly zero, because these are BPS states, which admit static multi-center configurations. One can, however, isolate the part corresponding to the exchange of R-R gauge bosons.

The force between two Dp -branes is computed in string perturbation theory by evaluating an annulus diagram with Dirichlet boundary conditions on both boundaries. This diagram can be viewed as a loop diagram of open strings or as a tree level diagram involving closed strings. In the later case the annulus is viewed as a cylinder connecting the two Dp -branes. For large separation this picture is more adequate, because the diagram is dominated by massless closed string states. There is an NS-NS contribution from the graviton and dilaton and an R-R contribution from the corresponding gauge field A_{p+1} . The R-R contribution gives a generalized Coulomb potential

$$V_{R-R}(r) = \frac{Q_{Dp}^2}{r^{D-p-3}} , \quad (4.66)$$

where the charge Q_{Dp} of a single Dp -brane is found to be

$$Q_{Dp} = g_S(\alpha')^{(7-p)/2} \frac{(2\pi)^{6-p}}{\omega_{6-p}} = c_p . \quad (4.67)$$

Thus a single Dp -brane carries precisely one unit of R-R p -brane charge. The long-distance R-R force between two R-R p -branes takes the same form, with charge $Q_p = \hat{Q}_p c_p$. This suggests that a R-R p -brane of charge Q_p is built out of \hat{Q}_p Dirichlet p -branes. Further tests of this idea can be performed by studying systems of p -branes and p' -branes, with $p \neq p'$. One can also consider the low velocity scattering of various fields on p -branes and Dp -branes.

The D -brane picture and the supergravity picture have different regimes of validity. The D -brane picture is valid within string perturbation theory. When considering a system of \hat{Q}_p D -branes, then the effective coupling is $\hat{Q}_p g_S$, because every boundary of the worldsheet can sit on any of the \hat{Q}_p D -branes. Therefore the validity of perturbation theory requires

$$\hat{Q}_p g_S \ll 1 . \quad (4.68)$$

In this picture the geometry is flat and D -branes are p -dimensional planes, which support an open string sector. Note that one is not restricted in energy. Using string perturbation theory it is possible to compute scattering processes at arbitrary energies, including those involving excited string states.

The range of validity of the supergravity picture is (almost) complementary. Here we have a non-trivial curved space-time which is an exact solution to the low energy equations of motion. In order to reliably use the low energy effective

action we are restricted to small curvature, measured in string units. According to (4.45) the Schwarzschild radius, which sets the scale of the solution is

$$r_S^{D-p-3} \simeq Q_{Dp} \simeq \hat{Q}_p g_S (\alpha')^{(D-p-3)/2} \quad (4.69)$$

Therefore the condition of weak curvature,²³

$$r_S \gg \sqrt{\alpha'} \quad (4.70)$$

is equivalent to

$$\hat{Q}_p g_S \gg 1, \quad (4.71)$$

which is the opposite of (4.68). The existence of a dual, perturbative description of R-R charged p -branes is a consequence of the particular dependence of the Schwarzschild radius on the string coupling g_S . According to (4.69) the Schwarzschild radius goes to zero relative to the string scale $\sqrt{\alpha'}$ when taking the limit $g_S \rightarrow 0$. This regime is outside the range of the supergravity picture, but inside the regime of perturbation theory, because

$$r_S \ll \sqrt{\alpha'} \Leftrightarrow \hat{Q}_p g_S \ll 1. \quad (4.72)$$

This relation also tells us why D -branes do not curve space-time, although they are dynamical objects which carry charge and mass: in the perturbative regime gravitational effects become arbitrarily small because the Schwarzschild radius is much smaller than the string scale.

This is a particular feature of R-R charged p -branes. One can regard it as a consequence of the particular dependence of the tension on the string coupling,

$$r_S^{D-p-3} \simeq G_N^{(D)} T_p \simeq g_S^2 \frac{1}{g_S} = g_S \rightarrow_{g_S \rightarrow 0} 0. \quad (4.73)$$

A completely different behaviour is exhibited by standard solitons like the NS-five-brane. Here the gravitational effects stay finite irrespective of the value of the coupling, because the tension goes like g_S^{-2} :

$$r_S^{D-8} \simeq G_N^{(10)} T_{NS5} \simeq g_S^2 \frac{1}{g_S^2} = 1. \quad (4.74)$$

Therefore there is no perturbative description of the NS-five-brane as a hyper-surface defect in flat space-time. NS-five-branes as seen by strings always have a non-trivial geometry.

Since the Dp -brane and the supergravity p -brane describe the same BPS state we can interpolate between the two pictures by varying g_S while keeping

²³For $p \neq 3$ the R-R p -brane solutions have a null singularity. Therefore the supergravity picture is only valid as long as one keeps a sufficient distance from the horizon. The R-R 3-brane has a regular event horizon and the same is true for certain more complicated configurations of intersecting branes, as we will see in the next section. For such objects the supergravity description is valid (at least) up to the event horizon. The same is true if one makes the R-R p -branes with $p \neq 3$ non-extremal, because in this case they have regular horizons with a curvature controlled by r_S .

\widehat{Q}_p fixed. We have no explicit description of the intermediate regime, but we can compute quantities in one picture and extrapolate to the other. This is used when counting black hole microstates using D -branes.

Before turning to this topic, we would like to digress and shortly explain how the Maldacena conjecture or AdS-CFT correspondence fits into the picture.

4.7 The AdS-CFT correspondence

The most simple example of the AdS-CFT correspondence is provided by a system of $N := \widehat{Q}_3$ $D3$ -branes. The crucial observation of Maldacena was that there is a regime in parameter space where both the D -brane picture and the supergravity picture apply. Let us start with the D -brane picture: Since gravity is an irrelevant interaction in $3 + 1$ dimensions, one can decouple it from the world volume theory of the $D3$ -brane by taking the low energy limit

$$\alpha' \rightarrow 0, \quad \text{with } g_S N \text{ and } \frac{R}{\alpha'} \text{ fixed} \quad , \quad (4.75)$$

where R is the typical scale of separation of the branes. The resulting world volume theory is four-dimensional $N = 4$ super-Yang-Mills with gauge group $U(N)$. R/α' is the typical mass scale, the W-mass. The gauge coupling is $g_{YM}^2 = 4\pi g_S$. After taking the low energy limit one can go to large Yang-Mills coupling. This regime is of course beyond perturbation theory, but since gravity and all other stringy modes have been decoupled, we know that it is still the same super Yang-Mills theory. Besides the strong coupling limit, $\lambda = Ng_{YM}^2 \gg 1$ one can consider the 't Hooft limit $N \rightarrow \infty$ with λ fixed.

On the supergravity side the low energy limit (4.75) is a near horizon limit which maps the whole solution onto its near horizon asymptotics. The $D3$ -brane has a smooth horizon, with geometry $AdS^5 \times S^5$. The supergravity picture is valid if the curvature is small, $Ng_S \gg 1$ or $\widehat{Q}g_{YM}^2 \gg 1$. In other words supergravity is valid in the strong coupling limit of the gauge theory, whereas $\frac{1}{N}$ corrections correspond to perturbative string corrections. This is the most simple example in a series of newly proposed 'bulk - boundary' dualities between supergravity or superstring theory on a nontrivial space-time and gauge theory on its (suitably defined) boundary. Since most of the cases considered so far relate supergravity or string theory on AdS space to conformally invariant gauge theories on its boundary, this is called the AdS - CFT correspondence. We will not enter into this subject here and refer the interested reader to Zaffaroni's lectures and to the literature.

4.8 Literature

An extensive introduction to string theory, which also covers D -branes and other more recently discovered aspects, is provided by Polchinski's books [10]. D -branes were also discussed in Gaberdiel's lectures at the TMR school. T-duality is reviewed by Giveon, Porrati and Rabinovici in [13]. The T-duality rules for R-R fields, which we did not write down, can be found in the paper [14]

by Bergshoeff and de Roo. Our discussion of p -branes follows Maldacena's thesis [12], where more details and references can be found. For a detailed account on p -branes in supergravity and string theory we also refer to the reviews by Duff, Khuri and Lu [15], Stelle [16] and Townsend [9]. The AdS - CFT correspondence was discussed in Zaffaroni's lectures at the TMR school. Finally we mentioned that $D(-1)$ -branes can be used to describe instantons in string theory. This was one of the subjects in Vandoren's lectures.

5 Black holes from p -branes

The basic idea of the D -brane approach to black hole entropy is the following: First one constructs extremal black holes by dimensional reduction of p -branes. This provides an embedding of such black holes into higher-dimensional supergravity and string theory. Second one uses the D -brane description of the p -branes to identify and count the states and to compute the statistical entropy $S_{\text{stat}} = \log N$. The result can then be compared to the Bekenstein-Hawking entropy $S_{BH} = \frac{A}{4}$ of the black hole.

This has been worked out in great detail over the last years for four- and five-dimensional extremal black holes in string compactifications with $N = 8, 4, 2$ supersymmetry. For simplicity we will consider the most simple case, extremal black holes in five-dimensional $N = 8$ supergravity. This is realized by compactifying type II string theory on T^5 . Before we can study this example, we have to explain how the dimensional reduction of the effective action and of its p -brane solutions works.

5.1 Dimensional reduction of the effective action

We illustrate the dimensional reduction of actions by considering the terms which are the most important for our purposes. The starting point is the string frame graviton - dilaton action in D dimensions,

$$S = \frac{1}{2\kappa_D^2} \int d^D x \sqrt{-G} e^{-2\phi_D} (R + 4\partial_M \phi_D \partial^M \phi_D) . \quad (5.1)$$

We take one direction to be periodic:

$$(x^M) = (x^\mu, x) , \text{ where } x \simeq x + 2\pi R . \quad (5.2)$$

The following decomposition of the metric leads directly to a $(D-1)$ -dimensional string frame action:

$$(G_{MN}) = \begin{pmatrix} \overline{G}_{\mu\nu} + e^{2\sigma} A_\mu A_\nu & e^{2\sigma} A_\mu \\ e^{2\sigma} A_\nu & e^{2\sigma} \end{pmatrix} , \quad (5.3)$$

where $\overline{G}_{\mu\nu}$ is the $(D-1)$ -dimensional string frame metric, A_μ is the Kaluza-Klein gauge field and σ is the Kaluza-Klein scalar. Observe that the decomposition

of G_{MN} is such that

$$\sqrt{-G} = e^\sigma \sqrt{-\bar{G}}. \quad (5.4)$$

Therefore the geodesic length $2\pi\rho$ of the internal circle and its parametric length $2\pi R$ are related by

$$2\pi\rho = 2\pi R e^{\langle\sigma\rangle}. \quad (5.5)$$

The vacuum expectation value $\langle\sigma\rangle$ of the Kaluza-Klein scalar is not fixed by the equations of motion and therefore $\langle\sigma\rangle$ is a free parameter characterizing the Kaluza-Klein vacuum. Such scalars are called moduli. Since the dilaton shows the same behaviour it is often also called a modulus.

One should note that only the combination $\rho = R e^{\langle\sigma\rangle}$ has an invariant meaning, because it is the measurable, geodesic radius of the internal circle. One has several options of parametrizing the compactification. One choice is to set $R = 1$ ($R = \sqrt{\alpha'}$ when restoring units) and to use $\langle\sigma\rangle$ to parametrize ρ . The other option is to redefine σ such that $\langle\sigma\rangle = 0$. Then the parametric and geodesic length are the same, $\rho = R$. As we have seen above similar remarks apply to the dilaton, which appears in the particular combination $\kappa_D e^{\phi_D}$ with the dimensionful string coupling κ_D . In the following we will use the convention that the vacuum expectation values of the geometric moduli and of the dilaton are absorbed in the corresponding parameters.

In order to get a $(D-1)$ -dimensional string frame action it is necessary to define the $(D-1)$ -dimensional dilaton by

$$\bar{\phi}_{D-1} = \phi_D - \frac{\sigma}{2}. \quad (5.6)$$

One now makes a Fourier expansion of the D -dimensional action (5.1) and drops the non-constant modes which describe massive modes from the $(D-1)$ -dimensional point of view. The resulting action is

$$\begin{aligned} S = & \frac{1}{2\kappa_{D-1}^2} \int d^{D-1}x \sqrt{-\bar{G}} e^{-2\bar{\phi}_{D-1}} (\bar{R} + 4\partial_\mu \bar{\phi}_{D-1} \partial^\mu \bar{\phi}_{D-1} - \partial_\mu \sigma \partial^\mu \sigma \\ & - \frac{1}{4} e^{2\sigma} F_{\mu\nu} F^{\mu\nu}), \end{aligned} \quad (5.7)$$

where the $(D-1)$ -dimensional coupling is

$$\frac{1}{\kappa_{D-1}^2} = \frac{2\pi R}{\kappa_D^2}. \quad (5.8)$$

Upon compactification the diffeomorphism invariance of the circle has turned into a $U(1)$ gauge symmetry. Massive states with a non-vanishing momentum along the circle are charged under this $U(1)$. Since the circle is compact, the charge spectrum is discrete and is of the form $Q \sim \frac{n}{R}$, with $n \in \mathbb{Z}$. Note that the gauge coupling in (5.7) is field dependent, and depends on both the dilaton and the modulus.

To reduce the full type IIA/B supergravity action one also has to consider the tensor fields B_{MN} and A_p and the fermions. We will not consider the fermionic terms here. Concerning the various p -form fields we remark that the reduction of a p -form on S^1 gives a p -form and a $(p-1)$ -form. Often one uses Hodge-duality to convert a p -form into a $(D-1-p)$ -form, in particular if $D-1-p < p$, because one wants to collect all terms with the same Lorentz structure. For example in $D=4$ the $B_{\mu\nu}$ field is dualized into the universal stringy axion, whereas in $D=5$ it is dualized into a gauge field.

5.2 Dimensional reduction of p -branes

There are two different ways of dimensionally reducing p -branes. The first and more obvious way is called double dimensional reduction or wrapping. Here one compactifies along a world volume direction of the brane. The reduction of a p -brane in D dimensions yields a $(p-1)$ -brane in $D-1$ dimensions, which is wrapped on the internal circle.

The second way is called simple dimensional reduction. This time one compactifies a transverse direction and obtains a p -brane in $D-1$ dimensions. Transverse directions are of course not isometry directions, but here we can make use of the no-force property of BPS branes. The idea is to first construct a periodic array of p -branes along one of the transverse directions and then to compactify.

To be specific let us split the transverse directions as $\vec{x} = (\underline{x}, x)$. Then we form a periodic array, such that the n -th p -brane sits at $x = 2\pi nR$, where $n \in \mathbb{Z}$. This array corresponds to the multi-center harmonic function

$$H = 1 + \sum_{n=-\infty}^n \frac{Q}{|\vec{x} - \vec{x}_n|^{D-p-3}}, \quad (5.9)$$

where $\vec{x}_n = (\underline{x}, 2\pi nR)$. Finally we impose the periodic identification $x \simeq x + 2\pi R$ and expand the harmonic function in Fourier modes

$$H = 1 + \frac{Q}{R |\underline{x}|^{D-p-4}} + O\left(e^{-|\underline{x}|/R}\right). \quad (5.10)$$

Since the non-constant Fourier modes are exponentially suppressed for small R , one can neglect them.

There is an alternative view of this procedure. The function

$$H = 1 + \frac{Q}{R |\underline{x}|^{D-p-4}} \quad (5.11)$$

is a spherically symmetric harmonic function with respect to the transverse Laplacian $\Delta_{\underline{x}}^\perp$ in $D-1$ dimensions. At the same time it is a harmonic function of the D -dimensional transverse Laplacian $\Delta_{\vec{x}}^\perp$ and therefore corresponds to a supersymmetric solution of the D -dimensional field equations. It has cylindrical rather than spherical symmetry and describes, from the D -dimensional point of

view, a p -brane which has been continuously smeared out along the x -direction. This might be viewed as the continuum limit of the periodic array discussed above. Such solutions are called delocalized p -branes. Since the direction in which the brane has been smeared out has become an isometry direction, one can compactify it.

5.3 The Tangherlini black hole

For simplicity we will construct five-dimensional black holes rather than four-dimensional ones. It is useful to know in advance how the five-dimensional analogue of the extreme Reissner-Nordstrom black hole looks like. This is the Tangherlini solution, which has the Einstein frame metric

$$ds_E^2 = -H^{-2}dt^2 + H(dr^2 + r^2 d\Omega_3^2) . \quad (5.12)$$

H is harmonic with respect to the four transverse directions. The single center function is

$$H = 1 + \frac{Q}{r^2} , \quad (5.13)$$

where Q is the electric charge. The solution is similar to the four-dimensional case, but with different powers of the harmonic function. We are using isotropic coordinates, and the event horizon is at $r = 0$. Its area is

$$A = 2\pi^2 \lim_{r \rightarrow 0} (r^3 H^{3/2}) = 2\pi^2 Q^{3/2} . \quad (5.14)$$

Here $2\pi^2$ is the area of the three-dimensional unit-sphere. With a trivial harmonic function, $Q = 0$, one gets flat space and the origin $r = 0$ is just a point. But for $Q \neq 0$ the metric is non-trivial, and $r = 0$ is a three-sphere.

The Bekenstein-Hawking entropy of this black hole is

$$S_{BH} = \frac{A}{4} = \frac{\pi^2}{2} Q^{3/2} . \quad (5.15)$$

5.4 Dimensional reduction of the $D1$ -brane

Let us now try to construct a five-dimensional black hole by dimensional reduction of the $D1$ -brane of type IIB on a five-torus T^5 . To fix notation we start with the ten-dimensional string frame metric

$$ds_{Str}^2 = H_1^{-1/2}(dt^2 + dy^2) + H_1^{1/2}(dx_1^2 + \cdots dx_8^2) \quad (5.16)$$

and ten-dimensional dilaton

$$e^{-2\phi_{10}} = H_1^{-1} . \quad (5.17)$$

The harmonic function is

$$H_1 = 1 + \frac{Q_1^{(10)}}{r^6} , \quad (5.18)$$

where $Q_1^{(10)}$ is the charge corresponding to the R-R two-form in the ten-dimensional IIB action. We now compactify the directions $x_5, \dots, x_8, x_9 = y$. The resulting string frame action is

$$ds_{Str}^2 = -H_1^{-1/2} dt^2 + H_1^{1/2} (dx_1^2 + \dots dx_4^2) , \quad (5.19)$$

where

$$H_1 = 1 + \frac{Q_1}{r^2} . \quad (5.20)$$

Here $Q_1 := Q_1^{(5)}$ is the charge with respect to the five-dimensional gauge field that is obtained by dimensional reduction of the ten-dimensional R-R two-form. As we know from our previous discussion, the gauge kinetic terms get dressed with dilaton and moduli dependent factors upon dimensional reduction. Thus $Q^{(5)} \neq Q_1^{(10)}$, and in order to know the precise expression for $Q_1 = Q_1^{(5)}$ one needs to carefully keep track of all the factors. We will give the explicit formula below.

The five-dimensional dilaton is

$$e^{-2\phi_5} = e^{-2\phi_{10}} \sqrt{G_{\text{internal}}} = H_1^{-1/4} . \quad (5.21)$$

To compute the Bekenstein-Hawking entropy we convert to the Einstein frame,

$$ds_E^2 = H_1^{-2/3} dt^2 + H_1^{1/3} (dr^2 + r^2 d\Omega_3^2) . \quad (5.22)$$

The area of the event horizon is

$$A = 2\pi^2 \lim_{r \rightarrow 0} \left(r^3 \sqrt{\frac{Q_1}{r^2}} \right) = 0 , \quad (5.23)$$

and therefore the Bekenstein-Hawking entropy vanishes:

$$S_{BH} = \frac{A}{4} = 0 . \quad (5.24)$$

The solution is degenerate: it does not have a finite event horizon. Instead we encounter a null singularity as for the p -brane solutions discussed above. This happens very often when constructing space-times with event horizons in the presence of non-trivial scalars. The scalars tend to take singular values at infinity or at the horizon and the geometry is affected by this. In our case the dilaton is singular at infinity and at the horizon. In the context of Kaluza-Klein theories such singularities are sometimes resolved by decompactification. This means that the singular values of the moduli at the horizon or at infinity indicate that the solution does not make sense as a solution of the lower dimensional theory. In our case we indeed observe that all internal radii either go to zero or to infinity at the horizon,

$$R_{5,6,7,8} \rightarrow \infty , \quad R_9 \rightarrow 0 . \quad (5.25)$$

In some cases (generically when the higher-dimensional solution does not have scalars) the decompactified higher-dimensional solution is regular at the horizon. This happens for example when considering the fundamental IIA string as a wrapped M2-brane of eleven-dimensional supergravity. In our case the decompactified solution is the $D1$ -brane, which is still singular. As discussed above, the singularity is interpreted in terms of a source.

We are interested in finding solutions with a regular metric and regular scalars. Both is correlated: Solutions with regular scalars usually have regular horizons. The problem of finding solutions with regular scalars is called the problem of 'stabilizing the moduli'. The generic method to achieve this is to construct solutions where the scalars are given by ratios of harmonic functions. Obviously one needs more than one harmonic functions in order to have non-constant scalars. In terms of D -branes this is realized by considering BPS superpositions of different types of Dp -branes.

Exercise XIV : Compactify the $D1$ -brane on T^5 and check the formulae given in this section.

5.5 Dp -brane superpositions

We are already familiar with the fact that BPS states admit multi-center realizations. More generally one can also find superpositions of different kinds of BPS states which still preserve part of the supersymmetry and are BPS states themselves. In order to find and classify these states one has to find configurations where the conditions on the Killing spinors of both types of BPS solutions are compatible.

We will need the special case of Dp - Dp' superpositions where the branes are either parallel or have rectangular intersections. In this case one gets BPS states if the number n of relative transverse dimensions is a multiple of 4,

$$n = 4k, \quad k \in \mathbb{Z}. \quad (5.26)$$

The relative transverse directions are those where one has Neumann boundary conditions with respect to one brane and Dirichlet boundary conditions with respect to the other.

Moreover the resulting state preserves $1/2$ of the supersymmetry for $k = 0$ and $1/4$ for $k = 1$. We need to consider the second case, which can be realized by a $D1$ -brane inside a $D5$ -brane, where we wrap all world-volume directions of the $D5$ -brane on the torus.

In the ten-dimensional string frame the metric of the $D1$ - $D5$ superposition is

$$\begin{aligned} ds_{Str}^2 = & H_1^{-1/2} H_5^{-1/2} (-dt^2 + dy_5^2) + H_1^{1/2} H_5^{1/2} (dx_1^2 + \cdots dx_4^2) \\ & + H_1^{1/2} H_5^{-1/2} (dy_1^2 + \cdots + dy_4^2), \end{aligned} \quad (5.27)$$

with dilaton

$$e^{-2\phi_{10}} = \frac{H_5}{H_1} . \quad (5.28)$$

Here t, y_5 are the overall parallel directions, y_1, \dots, y_4 the relative transverse directions and x_1, \dots, x_4 the overall transverse directions. When taking H_1 (H_5) to be trivial, we get back the $D5$ ($D1$) solution. Thus these solutions take the form of superpositions, despite that they solve non-linear equations of motion. The solution has 8 Killing spinors and preserves 1/4 of the supersymmetries. The conditions on the Killing spinors are

$$\varepsilon_1 = \Gamma^0 \Gamma^9 \varepsilon_2 \text{ and } \varepsilon_1 = \Gamma^0 \Gamma^5 \dots \Gamma^9 \varepsilon_2 , \quad (5.29)$$

where the labeling of directions is given according to $(x_0, x_1, \dots, x_4, x_5 = y_1, \dots, x_9 = y_5)$. The first condition is associated with the $D1$ -brane, the second with the $D5$ -brane. Every condition fixes half of the supersymmetry transformation parameters in terms of the other half and when combining them only 1/4 of the parameters are independent. Note that the ten-dimensional spinors $\varepsilon_{1,2}$ have the same chirality, since we are in the chiral IIB theory.

After dimensional reduction on T^5 the Einstein metric is found to be

$$ds_E^2 = -(H_1 H_5)^{-2/3} dt^2 + (H_1 H_5)^{1/3} (dr^2 + r^2 d\Omega_3^2) . \quad (5.30)$$

The area of the event horizon is

$$A = 2\pi^2 \lim_{r \rightarrow 0} \left(r^3 \sqrt{\frac{Q_1 Q_5}{r^4}} \right) = 0 \quad (5.31)$$

and therefore the Bekenstein-Hawking entropy is still zero. In order to find a regular solution we have to construct a BPS superposition with yet another object.

5.6 Superposition of $D1$ -brane, $D5$ -brane and pp-wave

One way of interpreting the vanishing entropy for the $D1$ - $D5$ system is that one is looking at the ground state of this system which is probably unique and therefore has zero entropy. Then we should look at excited BPS states of the system. One possibility is to add momentum along the $D1$ -brane. In the supergravity picture this is realized by superimposing a gravitational wave. More precisely the gravitational waves we have to consider are planar fronted gravitational waves with parallel rays, or pp-waves for short. They are purely gravitational solutions of the equations of motion and do not carry charge under the tensor fields. Instead they carry left- or right-moving momentum. Waves with purely left- or right-moving momentum are 1/2 BPS states.²⁴

²⁴pp-waves are further discussed in the exercises XIII and XV.

We now consider a superposition of $D1$ -brane, $D5$ -brane and a left-moving pp-wave along the $D1$ -brane. The ten-dimensional string frame metric is

$$ds_{Str}^2 = (H_1 H_5)^{-1/2} (-dt^2 + dy_5^2 + (H_K - 1)(dt^2 - dy_5^2)) \\ + (H_1 H_5)^{1/2} (dx_1^2 + \dots + dx_4^2) + H_1^{1/2} H_5^{-1/2} (dy_1^2 + \dots + dy_4^2) \quad (5.32)$$

and the dilaton is

$$e^{-2\phi_{10}} = \frac{H_5}{H_1} . \quad (5.33)$$

The metric for a pp-wave is obtained by taking $H_1 = 1 = H_5$. It depends on a harmonic function H_K . The presence of the pp-wave imposes the additional conditions

$$\Gamma^0 \Gamma^9 \varepsilon_1 = \varepsilon_1 \text{ and } \Gamma^0 \Gamma^9 \varepsilon_2 = \varepsilon_2 \quad (5.34)$$

on the Killing spinors. As a consequence the resulting configuration has four Killing spinors and preserves 1/8 of the supersymmetries of the vacuum.

After dimensional reduction on T^5 we obtain the following Einstein frame metric:

$$ds_{Str}^2 = -(H_1 H_5 H_K)^{-2/3} dt^2 + (H_1 H_5 H_K)^{1/3} (dr^2 + r^2 d\Omega_3^2) . \quad (5.35)$$

The harmonic functions are

$$H_i = 1 + \frac{Q_i}{r^2} , \quad i = 1, 5, K , \quad (5.36)$$

where Q_1, Q_5 are the five-dimensional charges of the gauge fields obtained by dimensional reduction of the R-R two-form. The corresponding ten-dimensional charges are the electric and the magnetic charge of the R-R two-form. In five dimensions electric charges are carried by zero-branes and magnetic charges are carried by one-branes. Therefore the five-dimensional charges Q_1, Q_5 are both electric. What happens is that the R-R two-form A_{MN} gives both one- and two-forms upon dimensional reduction. However in five dimensions the two-forms can be Hodge dualized into one-forms, and these are the objects which couple locally to zero-branes. (The reduction of the two-form on T^5 gives of course several one- and two-forms, but most of them are trivial in the solution we consider.) The parameter Q_K is related to the momentum of the pp-wave around the y_5 -direction. From the lower dimensional point of view this is the charge with respect to one of the Kaluza-Klein gauge fields. All three kinds of charges are integer multiples of unit charges, $Q_i = \hat{Q}_i c_i$, where $\hat{Q}_i \in \mathbb{Z}$. The five-dimensional unit charges are

$$c_1 = \frac{4G_N^{(5)} R_9}{\pi \alpha' g_S} , \quad c_5 = g_S \alpha' , \quad c_K = \frac{4G_N^{(5)}}{\pi R_9} . \quad (5.37)$$

Here $G_N^{(5)}$ is the five-dimensional Newton constant, R_9 the radius of the y_5 -direction and g_S is the ten-dimensional string coupling.²⁵ The non-trivial scalar fields of the solution are the five-dimensional dilaton and the Kaluza-Klein scalar σ , which parametrizes the volume of T^5 :

$$e^{-2\phi_5} = \frac{H_K^{1/2}}{(H_1 H_5)^{1/4}}, \quad e^{-2\sigma} = \left(\frac{H_1}{H_5} \right)^{1/2}. \quad (5.38)$$

Both scalars are given by ratios of harmonic functions and are finite throughout the solution. The Einstein frame metric is

$$ds_E^2 = -(H_1 H_5 H_K)^{-2/3} dt^2 + (H_1 H_5 H_K)^{1/3} (dr^2 + r^2 d\Omega_3^2). \quad (5.39)$$

This metric is regular at the horizon and the near horizon geometry is $AdS^2 \times S^3$. The area is

$$A = 2\pi^2 \lim_{r \rightarrow 0} \left(r^3 \sqrt{\frac{Q_1 Q_5 Q_K}{r^6}} \right) = 2\pi^2 \sqrt{Q_1 Q_5 Q_K}. \quad (5.40)$$

Thus we now get a finite Bekenstein-Hawking entropy. It is instructive to restore dimensions and to express the entropy in terms of the integers \hat{Q}_i :

$$S_{BH} = \frac{A}{4G_N^{(5)}} = \frac{\pi^2}{2G_N^{(5)}} \sqrt{Q_1 Q_5 Q_K} = 2\pi \sqrt{\hat{Q}_1 \hat{Q}_5 \hat{Q}_K}. \quad (5.41)$$

All dimensionful constants and all continuous parameters cancel precisely and the entropy is a pure number, which is given by the numbers of $D1$ -branes, $D5$ -branes and quanta of momentum along the $D1$ -branes. This indicates that an interpretation in terms of microscopic D -brane states is possible. We will come to this later.

In contrast to the entropy the mass depends on both the charges and on the moduli. The minimum of the mass as a function of the moduli is obtained when taking the scalar fields to be constant,

$$e^{-2\phi_5} = 1 \text{ and } e^{-2\sigma} = 1. \quad (5.42)$$

This amounts to equating all the charges and all the harmonic functions:

$$Q = Q_1 = Q_5 = Q_K, \quad H = H_i = 1 + \frac{Q}{r^2} \quad (5.43)$$

The resulting solution is precisely the Tangherlini solution. Solutions with constant scalars are called double extreme. They can be deformed into generic extreme solutions by changing the values of the moduli at infinity. It turns out that the values of the moduli at the horizon cannot change, but are completely

²⁵To derive this one has to carefully carry out the dimensional reduction of the action. Of course there is a certain conventional arbitrariness in normalizing gauge fields and charges. We use the conventions of [12].

fixed in terms of the charges of the black hole. This is referred to as fixed point behaviour. The origin of this behaviour is that the stabilization of the moduli (i.e. a regular solution) is achieved through supersymmetry enhancement at the horizon: whereas the bulk solution has four Killings spinors, the asymptotic solution on the event horizon has eight. The values of the scalars at the horizon have to satisfy relations, called the stabilization equations, which fix them in terms of the charges. This has been called the supersymmetric attractor mechanism, because the values of the scalars are arbitrary at infinity but are attracted to their fixed point values when going to the horizon. The geometry of the near horizon solution is $AdS^2 \times S^3$.

Exercise XV : Compactify the D -dimensional pp-wave, $D > 4$,

$$ds_E^2 = (K - 1)dt^2 + (K + 1)dy^2 - 2Kdydt + d\vec{x}^2 \quad (5.44)$$

over y . Take the harmonic function to be $K = \frac{Q}{r^{D-4}}$. Why did we delocalize the solution along y ? What happens for $D = 4$? What is the interpretation of the parameter Q from the $(D - 1)$ -dimensional point of view?

5.7 Black hole entropy from state counting

We now have to analyse the black hole solution in the D -brane picture in order to identify and count its microstates and to compute the statistical entropy. The D -brane configuration consists of \widehat{Q}_5 $D5$ -branes and \widehat{Q}_1 $D1$ -branes wrapped on T^5 . Moreover \widehat{Q}_K quanta of light-like, left-moving (for definiteness) momentum have been put on the $D1$ -branes. This is an excited BPS state and the statistical entropy counts in how many different ways one can distribute the total momentum between the excitations of the system. Since the momentum is light-like, we have to look for the massless excitations. We can perform the counting in the corner of the parameter space which is most convenient for us, because we are considering a BPS state.

In particular we can split the T^5 as $T^4 \times S^1$ and make the circle much larger than the T^4 . After dimensional reduction on T^4 the D -brane system is 1 + 1 dimensional, with compact space. At low energies the effective world volume theory of \widehat{Q}_p Dp -branes is a dimensionally reduced $U(\widehat{Q}_p)$ super Yang-Mills theory. In our case we get a two-dimensional Yang-Mills theory with $N = (4, 4)$ supersymmetry and gauge group $U(\widehat{Q}_1) \times U(\widehat{Q}_5)$. The corresponding excitations are the light modes of open strings which begin and end on $D1$ -branes or begin and end on $D5$ -branes. In addition there are open strings which connect $D1$ -branes to $D5$ -branes or vice versa. The light modes of these strings provide additional hypermultiplets in the representations $\widehat{Q}_1 \times \overline{\widehat{Q}}_5$ and $\widehat{Q}_5 \times \overline{\widehat{Q}}_1$, where \widehat{Q}_p and $\overline{\widehat{Q}}_p$ are the fundamental representation and its complex conjugate.

In order to identify the massless excitations of this theory, one has to find the flat directions of its scalar potential. The potential has a complicated valley structure. There are two main branches, called the Coulomb branch and the Higgs branch. The Coulomb branch is parametrized by vacuum expectation

values of scalars in vector multiplets, whereas the Higgs branch is parametrized by vacuum expectation values of scalars in hypermultiplets. Note that once a massless excitation along the Coulomb branch (Higgs branch) has been turned on, all excitations corresponding to fundamental (adjoint) scalars become massive and their vacuum expectation values have to vanish. The two kinds of massless excitations mutually exclude one another. Since we expect that the state with maximal entropy is realized, we have to find out which of the two branches has the higher dimension.

Along the Coulomb branch the gauge group is broken to the $U(1)^{\widehat{Q}_1} \times U(1)^{\widehat{Q}_5}$. The number of massless states is proportional to the number $\widehat{Q}_1 + \widehat{Q}_5$ of directions along the Cartan subalgebra. Geometrically, turning on vacuum expectation values of the adjoint scalars corresponds to moving all the $D1$ -branes and $D5$ -branes to different positions. Then only open strings which start and end on the same brane can have massless excitations. The state describing the black hole is expected to be a bound state, where all the branes sit on top of each other. This is not what we find in the Coulomb branch.

Along the Higgs branch the gauge group is broken to $U(1)$. Since the vacuum expectation values of all adjoint scalars, which encode the positions of the branes, are frozen to zero, all branes sit on top of each other and form a bound state, as expected for a black hole. The unbroken $U(1)$ corresponds to the overall translational degree of freedom of the bound state. A careful analysis shows that the potential has $4\widehat{Q}_1\widehat{Q}_5$ flat directions, corresponding to scalars in $\widehat{Q}_1\widehat{Q}_5$ hypermultiplets. The massless degrees of freedom are the $4\widehat{Q}_1\widehat{Q}_5$ scalars and the $2\widehat{Q}_1\widehat{Q}_5$ Weyl spinors sitting in these multiplets.

If we take the circle to be very large, then the energy carried by individual excitations is very small. Therefore we only need to know the IR limit of the effective theory of the massless modes. The $N = (4, 4)$ supersymmetry present in the system implies that the IR fixed point is a superconformal sigma-model with a hyper Kähler target space. Therefore the central charge can be computed as if the scalars and fermions were free fields. Since a real boson (a Majorana-Weyl fermion) carries central charge $c = 1$ ($c = \frac{1}{2}$), the total central charge is

$$c = \left(1 + \frac{1}{2}\right) 4\widehat{Q}_1\widehat{Q}_5 = 6\widehat{Q}_1\widehat{Q}_5 . \quad (5.45)$$

We now use Cardy's formula for the asymptotic number $N(E)$ of states in a two-dimensional conformal field theory with compact space:

$$N(E) = \exp S_{\text{Stat}} \simeq \exp \sqrt{\pi c E L / 3} , \quad (5.46)$$

where E is the total energy and L the volume of space. The formula is valid asymptotically for large E . Using that

$$E = \frac{|\widehat{Q}_K|}{R} \text{ and } L = 2\pi R , \quad (5.47)$$

we find

$$S_{\text{Stat}} = 2\pi \sqrt{\frac{c}{6} \widehat{Q}_K} = 2\pi \sqrt{\widehat{Q}_1 \widehat{Q}_5 \widehat{Q}_K} , \quad (5.48)$$

which is precisely the Bekenstein-Hawking entropy of the black hole.

5.8 Literature

Our treatment of dimensional reduction follows Maldacena's thesis [12] and the books of Polchinski [10] and Behrndt [17]. The derivation of the statistical entropy through counting of D -brane states is due to Strominger and Vafa [18]. The exposition given in the last section follows [12]. The $D1 - D5$ system has been the subject of intensive study since then, see for example [19] or [20] for recent reviews and references. The supersymmetric attractor mechanism was discovered by Ferrara, Kallosh and Strominger [21].

5.9 Concluding Remarks

With the end of these introductory lectures we have reached the starting point of the recent research work on black holes in the context of string theory. Let us briefly indicate some further results and give some more references.

The above example of a matching between the Bekenstein-Hawking entropy and the statistical entropy was for a five-dimensional black hole in $N = 8$ supergravity. This has been generalized to compactifications with less supersymmetry and to four-dimensional black holes. The most general set-up where extremal black holes can be BPS solitons are four-dimensional $N = 2$ compactifications. The Bekenstein-Hawking entropy for such black holes was found by Behrndt et al [22], whereas the corresponding state counting was performed by Maldacena, Strominger and Witten [23] and by Vafa [24]. To find agreement between the geometric entropy computed in four-dimensional $N = 2$ supergravity and the statistical entropy found by state counting one must properly include higher curvature terms on the supergravity side and replace the Bekenstein-Hawking area law by a refined definition of entropy, which is due to Wald and applies to gravity actions with higher curvature terms [25]. An upcoming paper of the author will provide a detailed review of black hole entropy in $N = 2$ compactifications, including the effects of higher curvature terms [26].

One aspect of black hole entropy is the dependence of the entropy on the charges. In the example discussed above the five-dimensional black hole carried three charges, whereas the most general extremal black hole solution in a five-dimensional $N = 8$ compactification carries 27 charges. It is of considerable interest to find the most general solution and the corresponding entropy, not only as a matter of principle but also because string theory predicts an invariance of the entropy formula under discrete duality transformations. Depending on the compactification these are called U-duality, S-duality or T-duality. In $N = 8$ and $N = 4$ compactifications these symmetries are exact and can be used to construct general BPS black hole solutions from a generating solution. Duality properties of entropy formulae and of solutions are reviewed by D'Auria and Fré [27]. More recently, a generating solution for regular BPS black holes in four-dimensional $N = 8$ compactifications has been constructed by Bertolini and

Trigiante, which allows for both a macroscopic and microscopic interpretation [34]. We refer to these works for more references on generating solutions.

In heterotic $N = 2$ compactifications T-duality is preserved in perturbation theory. Since the low energy effective action receives both loop and α' corrections, the situation is more complicated than in $N = 4, 8$ compactifications. Nevertheless one can show that the entropy is T -duality invariant and one can find explicit T -duality invariant entropy formulae in suitable limits in moduli space. This is discussed in [28] and will be reviewed in [26].

One can also construct explicit general multi-center BPS black hole solutions in four- and five-dimensional $N = 2$ supergravity, which are parametrized by harmonic functions and generalize the Majumdar-Papapetrou solutions and the stationary IWP solutions. These were found, for the four-dimensional case, by Sabra [29] and by Behrndt, Lüst and Sabra [8].

Another direction is to go away from the BPS limit and to study near-BPS states. The most prominent application is the derivation of Hawking radiation from the D -brane perspective. Near-BPS states can for example be described by adding a small admixture of right-moving momentum to a purely left-moving BPS state. In the D -brane picture left- and right-moving open strings can interact, form closed string states and leave the brane. The resulting spectrum quantitatively agrees, when averaging over initial and summing over final states, with thermal Hawking radiation. Hawking radiation and other aspects of near-extremal black holes are reviewed in [12], [19], [20] and [33].

A second way to go away from the BPS limit is to deform multi-center BPS solutions by giving the black holes a small velocity. To leading order, such systems are completely determined by the metric on the moduli space of multi-center solutions. The quantum dynamics of such a system, including interactions of black holes can be studied in terms of quantum mechanics on the moduli space. In the so-called near horizon limit the quantum dynamics becomes superconformal and it seems possible to learn about black hole entropy in terms of bound states of the conformal Hamiltonian. This subject is reviewed in [30].

Finally we would like to point out that there are other approaches to black holes in string theory. An older idea is the identification of black holes with excited elementary string states. This can be formulated in terms of a correspondence principle and is reviewed in [10]. Schwarzschild black holes have been discussed using the Matrix formulation of M-theory [31]. Another approach is to map four- and five-dimensional black holes to three-dimensional black holes through T-duality transformations which are asymptotically light-like. One then uses that three-dimensional gravity has no local degree of freedoms, and counts microstates in a two-dimensional conformal field theory living on the boundary of space-time. This is for example reviewed in [32]. Finally the development of the AdS-CFT correspondence is closely interrelated with various aspects of black hole physics [33]. All these approaches are not tied to BPS states. This opens the perspective that a satisfactory quantitative understanding of non-supersymmetric black holes will be achieved in the future.

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A Solutions of the exercises

Solution I : For a radial lightray we have $ds^2 = 0$ and $d\theta = 0 = d\phi$. Thus:

$$ds^2 = -\left(1 - \frac{r_S}{r}\right) dt^2 + \left(1 - \frac{r_S}{r}\right)^{-1} dr^2 = 0 \quad (\text{A.1})$$

or

$$\left(\frac{dt}{dr}\right)^2 = \left(1 - \frac{r_S}{r}\right)^{-2}. \quad (\text{A.2})$$

We take the square root:

$$\frac{dt}{dr} = \pm \left(1 - \frac{r_S}{r}\right)^{-1}. \quad (\text{A.3})$$

The + sign corresponds to outgoing lightrays, the - sign to ingoing ones. By integration we find the time intervall

$$\Delta t = t_2 - t_1 = \pm \left[(r_2 - r_1) + r_S \log \frac{r_2 - r_S}{r_1 - r_S} \right]. \quad (\text{A.4})$$

In the limit $r_1 \rightarrow r_S$ the time intervall diverges, $\Delta t \rightarrow \infty$.

Solution II : The frequency ω_i^μ is given by the time component of the four-momentum k^μ with respect to the frame of a static observer at r_i . The time-direction of this frame is defined by the four-velocity u_i^μ , which is the normalized tangent to the world line, $u_i^\mu u_{\mu i} = -1$. Therefore the frequency is

$$\omega_i = -k_\mu u_i^\mu. \quad (\text{A.5})$$

The sign is chosen such that ω_i is positive when k^μ, u_i^μ are in the forward light cone.

Next we prove equation (2.18). t^μ is the tangent of a geodesic and ξ^μ is a Killing vector field. The product rule implies:

$$t^\mu \nabla_\mu (\xi_\nu t^\nu) = t^\mu (\nabla_\mu \xi_\nu) t^\nu + t^\mu \xi_\nu \nabla_\mu t^\nu. \quad (\text{A.6})$$

The first term vanishes by the Killing equation, $\nabla_{(\mu}\xi_{\nu)} = 0$, whereas the second term is zero as a consequence of the geodesic equation $t^\mu \nabla_\mu t^\nu = 0$.

In the context of our exercise the Killing vector field is the static Killing vector field $\xi = \frac{\partial}{\partial t}$ and the tangent vector is the four-momentum of the light ray, k^μ . The conservation law implies

$$(\xi_\nu k^\nu)_{r_1} = (\xi_\nu k^\nu)_{r_2} . \quad (\text{A.7})$$

The conserved quantity is the projection of the four-momentum on the direction defined by the timelike Killing vector field. Therefore it is interpreted as energy. More generally energy is conserved along geodesics if a metric has a timelike Killing vector.

Finally the vectors u_i^μ and ξ^μ are parallel: The observers at r_i are static which means that their time-directions coincide with the Schwarzschild time. The constant of proportionality is fixed by the normalization: By definition four-velocities are normalized as $u_i^\mu u_{\mu i} = -1$ whereas the norm of the Killing vector $\xi^\mu = (1, 0, 0, 0)$ is

$$\xi^\mu \xi_\mu = g_{\mu\nu} \xi^\mu \xi^\nu = g_{tt} = -\left(1 - \frac{r_S}{r}\right) . \quad (\text{A.8})$$

Therefore

$$\xi^\mu = V(r_i) u_i^\mu , \quad (\text{A.9})$$

where $V(r_i) = \sqrt{-g_{tt}}$ is the redshift factor.

Putting everything together we find

$$\frac{\omega_1}{\omega_2} = \frac{k_\mu u_1^\mu}{k_\mu u_2^\mu} = \frac{V(r_2)}{V(r_1)} \frac{k_\mu \xi^\mu}{k_\mu \xi^\mu} = \frac{V(r_2)}{V(r_1)} . \quad (\text{A.10})$$

Solution III : We use the relation between the four-acceleration a^μ , the four-velocity u^μ , the Killing vector ξ^μ and the redshift factor:

$$a^\mu = \frac{du^\mu}{d\tau} = u^\nu \nabla_\nu u^\mu = \frac{\xi^\nu}{V} \nabla_\nu \frac{\xi^\mu}{V} = \frac{\xi^\nu}{\sqrt{-\xi_\rho \xi^\rho}} \nabla_\nu \frac{\xi^\mu}{\sqrt{-\xi_\rho \xi^\rho}} . \quad (\text{A.11})$$

The formulae (2.19) and (2.20) follow by working out the derivatives and making use of the Killing equation $\nabla_{(\mu}\xi_{\nu)} = 0$.

To derive (2.23) we compute

$$\partial_r V = \partial_r \sqrt{-\xi_\nu \xi^\nu} = \frac{r_S}{2r^2 \sqrt{1 - \frac{r_S}{r}}} , \quad (\text{A.12})$$

$$\nabla_\mu V \nabla^\mu V = g^{rr} (\partial_r V)^2 = \frac{r_S^2}{4r^4} \quad (\text{A.13})$$

and finally find

$$\kappa_S = (Va)_{r=r_S} = \sqrt{\nabla_\mu V \nabla^\mu V} \Big|_{r=r_S} = \frac{r_S}{2r^2} \Big|_{r=r_S} = \frac{1}{2r_S} = \frac{1}{4M} . \quad (\text{A.14})$$

Solution IV :

If $F_{\mu\nu}$ is static and spherically symmetric, then the independent non-vanishing components are $F_{tr}(r, \theta, \phi)$ and $F_{\theta\phi}(r, \theta, \phi)$. Next we note that

$$\nabla_\mu F^{\mu\nu} = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} F^{\mu\nu}) . \quad (\text{A.15})$$

This identity is generally valid for the covariant divergence of antisymmetric tensors of arbitrary rank. For a metric of the form (2.32) we have

$$\sqrt{-g} = e^{g+f} r^2 \sin \theta . \quad (\text{A.16})$$

The only non-trivial equation of motion for the electric part is

$$\partial_r (\sqrt{-g} F^{rt}) = \partial_r (e^{g+f} r^2 \sin \theta \cdot (-e^{-2g-2f} F_{rt})) = \partial_r (e^{-g-f} r^2 \sin \theta F_{tr}) = 0 , \quad (\text{A.17})$$

which implies

$$F_{tr} = e^{g+f} \frac{q(\theta, \phi)}{r^2} . \quad (\text{A.18})$$

But we also have to impose the Bianchi identities $\varepsilon^{\mu\nu\rho\sigma} \partial_\nu F_{\rho\sigma} = 0$, which imply $\partial_\theta F_{tr} = 0 = \partial_\phi F_{tr}$ and therefore we have

$$F_{tr} = e^{g+f} \frac{q}{r^2} \quad (\text{A.19})$$

with a constant q .

The non-trivial equations of motion for the magnetic part are

$$\begin{aligned} \partial_\theta (e^{g+f} r^2 \sin \theta F^{\theta\phi}) &= 0 , \\ \partial_\phi (e^{g+f} r^2 \sin \theta F^{\phi\theta}) &= 0 , \end{aligned} \quad (\text{A.20})$$

which are solved by

$$F_{\theta\phi} = p(r) \sin \theta . \quad (\text{A.21})$$

Since the Bianchi identities imply $\partial_r F_{\theta\phi} = 0$ we finally have

$$F_{\theta\phi} = p \sin \theta , \quad (\text{A.22})$$

with constant p .

Solution V : Choose the integration surface to be a sphere $r = \text{const}$ with sufficiently large r . Introduce coordinates $y^\alpha = (\theta, \phi)$ on this sphere. Then

$$\oint {}^*F = \oint \frac{1}{2} {}^*F_{\mu\nu} dx^\mu \wedge dx^\nu = \oint \frac{1}{2} {}^*F_{\alpha\beta} dy^\alpha \wedge dy^\beta = \int_0^\pi d\theta \int_0^{2\pi} d\phi {}^*F_{\theta\phi} . \quad (\text{A.23})$$

Evaluate the \star -dual:

$$\star F_{\theta\phi} = \sqrt{-g} F^{tr} . \quad (\text{A.24})$$

(The Levi-Civita tensor $\varepsilon_{\mu\nu\rho\sigma}$ in the definition (2.27) contains a factor $\sqrt{-g}$.) Finally plug in F^{tr} and $\sqrt{-g}$:

$$\oint \star F = \int_0^\pi d\theta \int_0^{2\pi} d\phi e^{-f-g} \frac{q}{r^2} e^{f+g} r^2 \sin \theta = 4\pi q . \quad (\text{A.25})$$

The second integral is

$$\oint F = \int_0^\pi d\theta \int_0^{2\pi} F_{\theta\phi} = \int_0^\pi d\theta \int_0^{2\pi} p \sin \theta = 4\pi p . \quad (\text{A.26})$$

Solution VI : The fields F_{tr} and $F_{\theta\phi}$ look different, because we use a coordinate system where the tangent vectors to the coordinate lines are not normalized. To get a more symmetric form we can use the vielbein to convert the curved indices $\mu, \nu = t, r, \theta, \phi$ into flat indices $a, b = 0, 1, 2, 3$. The natural choice of the vielbein for a spherically symmetric metric (2.32) is

$$e_\mu^a = \text{diag}(e^g, e^f, r, r \sin \theta) , \quad e_a^\mu = \text{diag}(e^{-g}, e^{-f}, \frac{1}{r}, \frac{1}{r} \sin^{-1} \theta) . \quad (\text{A.27})$$

Now we compute $F_{ab} = e_a^\mu e_b^\nu F_{\mu\nu}$ we the result

$$F^{01} = \frac{q}{r^2} , \quad F^{23} = \frac{p}{r^2} . \quad (\text{A.28})$$

Thus when expressed using flat indices the gauge field looks like a static electric and magnetic point charge in flat space.

Solution VII : The coordinate transformation is $r \rightarrow r - M$. In the limit $r \rightarrow 0$ of (3.2) the (θ, ϕ) part of the metric is

$$\left(1 + \frac{M}{r}\right)^2 r^2 d\Omega^2 \xrightarrow{r \rightarrow 0} \frac{M^2}{r^2} r^2 d\Omega^2 = M d\Omega^2 . \quad (\text{A.29})$$

Thus integration over θ, ϕ at $r = 0$ and arbitrary t yields $A = 4\pi M^2$, which is the area of a sphere of radius M .

To show that (3.3) is conformally flat, we introduce a new coordinate ρ by $\rho/M = M/r$. Then the metric takes the form

$$ds^2 = \frac{M^2}{r^2} (-dt^2 + d\rho^2 + \rho^2 d\Omega^2) . \quad (\text{A.30})$$

This is manifestly conformally flat, because the expression in brackets is the flat metric, written in spheric coordinates.

Solution VIII : The non-trivial equations of motion for $F_{ti} = \mp \partial_i e^{-f} = F^{it}$ are

$$\partial_i (\sqrt{-g} F^{it}) = 0 , \quad (\text{A.31})$$

with $\sqrt{-g} = e^{2f}$, where $f = f(\vec{x})$. Plugging in the ansatz we get

$$\sum_i \partial_i (e^{2f} \partial_i e^{-f}) = \sum_i \partial_i \partial_i e^f = 0 . \quad (\text{A.32})$$

Thus e^f must be a harmonic function.

Solution IX : In order to one single equation for the background, the two terms must be linearly dependent. Up to a phase this leads to the ansatz

$$\epsilon_A = -\gamma^0 \varepsilon_{AB} \epsilon^B . \quad (\text{A.33})$$

We will comment on the significance of the phase later. Plugging this ansatz into the Killing equation we get

$$F_{0i}^- = \frac{1}{2} \partial_i e^{-f} . \quad (\text{A.34})$$

Since the right hand side is real it follows

$$F_{0i} = \partial_i e^{-f} \quad F_{ij} = 0 . \quad (\text{A.35})$$

This solution is purely electric. If we take a different phase in the ansatz, we get a dyonic solution instead. Finally the Maxwell equations imply that e^f must be harmonic,

$$\Delta e^f = 0 . \quad (\text{A.36})$$

This is precisely the Majumdar-Papapetrou solution.

Solution X : When decomposing the Majorana supercharges in term of Weyl spinors

$$Q_a^A = \begin{pmatrix} Q_\alpha^A \\ \bar{Q}^{A\dot{\alpha}} \end{pmatrix} , \quad (\text{A.37})$$

one finds that

$$\begin{pmatrix} \{Q_\alpha^1, Q_\beta^2\} & \{Q_\alpha^1, \bar{Q}_\beta^2\} \\ \{\bar{Q}^{2\dot{\alpha}}, Q_\beta^2\} & \{\bar{Q}^{2\dot{\alpha}}, \bar{Q}_\beta^2\} \end{pmatrix} \sim \gamma^\mu P_\mu + ip \mathbb{I} + q \gamma_5 . \quad (\text{A.38})$$

Solution XI : We have to solve

$$(\gamma^\mu P_\mu + ip + q\gamma_5)\epsilon = 0 . \quad (\text{A.39})$$

For a massive state at rest we have

$$\gamma^\mu P_\mu = -M\gamma^0 . \quad (\text{A.40})$$

Decomposing the Dirac spinor ϵ into chiral spinors,

$$\epsilon = \epsilon_+ + \epsilon_- \text{ where } \gamma_5 \epsilon_\pm = \pm \epsilon_\pm , \quad (\text{A.41})$$

we get

$$M\epsilon_- + i\gamma^0(p - iq)\epsilon_+ = 0 . \quad (\text{A.42})$$

Now we use that $Z = p - iq$ is the central charge. Moreover, for a BPS state we have $Z = M\frac{Z}{|Z|}$. This yields

$$\epsilon_- + i\gamma^0 \frac{Z}{|Z|} \epsilon_+ = 0 , \quad (\text{A.43})$$

which is the same projection that one finds when solving the Killing spinor equations for the extreme Reissner-Nordstrom black hole.

Solution XII : The covariant derivative is defined by

$$D_\pm X^1 = \partial_\pm X^1 + A_\pm , \quad (\text{A.44})$$

where the gauge field A_\pm transforms as

$$A_\pm \rightarrow A_\pm - \partial_\pm a \quad (\text{A.45})$$

under $X^1 \rightarrow X^1 + a(\sigma)$. The non-vanishing component of the corresponding field strength is $F_{+-} = \partial_+ A_- - \partial_- A_+$. In the locally invariant action (4.7) we have imposed that this field strength is trivial using a Lagrange multiplier \tilde{X}^1 . Using the equation of motion $F_{+-} = 0$ and choosing the gauge $A_\pm = 0$ we get back the original action (4.6). (We are ignoring global aspects.) Eliminating the gauge field through its equation of motion results in

$$S[\tilde{G}_{11}] = \int d^2\sigma \tilde{G}_{11} \partial_+ \tilde{X}^1 \partial_- \tilde{X}^1 , \quad (\text{A.46})$$

where $\tilde{G}_{11} = \frac{1}{G_{11}}$.

Thus, by the field redefinition $X^1 \rightarrow \tilde{X}^1$ we have inverted the target space metric along the isometry direction. If the 1-direction is compact this means that string theory on a circle with radius R is equivalent to string theory on a circle with the inverse radius R^{-1} . In the non-compact case small and large curvature are related. This is the simplest example of T-duality. In the example we have derived it from the world-sheet perspective, and for curved target spaces with isometries. This formulation is also known as Buscher duality.

Solution XIII : We first apply T-duality along the world-volume direction y . Using (4.8) we get

$$ds'^2_{Str} = (K - 1)dt^2 + (K + 1)dy^2 - 2Kdtdy + d\vec{x}^2, \quad (\text{A.47})$$

where $K = H_1 - 1$ and

$$\phi' = 0 \Rightarrow ds'^2_{Str} = ds'^2_E. \quad (\text{A.48})$$

The non-trivial B -field of the fundamental string has been transformed into an off-diagonal component of the metric. Moreover the dilaton is trivial in the new background, which therefore is purely gravitational. (A.47) is a special case of a gravitational wave (pp-wave). Introducing light-cone coordinates $u = y - t$ and $v = y + t$ one gets the standard form

$$ds'^2_E = dudv + Kdu^2 + d\vec{x}^2. \quad (\text{A.49})$$

When T-dualizing a single center fundamental string, the function K takes the form $K = \frac{Q_1}{r^6}$. More generally, pp-waves are 1/2 BPS states if K has an arbitrary dependence on u and is harmonic with respect to the transverse coordinates,

$$\Delta_{\vec{x}}K(u, \vec{x}) = 0. \quad (\text{A.50})$$

The solution has a lightlike Killing vector ∂_v . The supersymmetry charge carried by it is its left-moving (or right-moving) momentum. The non-trivial u -dependence can be used to modulate the amplitude of the wave. For the T-dual of the fundamental string the amplitude is constant and the total momentum is infinite unless one compactifies the y -direction.

When superimposing a pp-wave with non-trivial u -dependence on a fundamental string solution, one gets various oscillating string solutions which are 1/4 BPS solutions. These solutions are in one to one correspondence with 1/4 BPS states of the perturbative type IIA/B string.

Let us next apply T-duality orthogonal to the world volume. Since these directions are not isometry directions, a modification of the procedure is needed. One first delocalizes the string along one of the directions, say x_8 . This is done by dropping the dependence of the solution on the corresponding coordinate:

$$H_1(r) = 1 + \frac{Q_1}{r^6} \rightarrow H_1(r') = 1 + \frac{Q_1}{r'^5}, \quad (\text{A.51})$$

where

$$r = \sqrt{x_1^2 + \cdots + x_8^2} \quad \text{and} \quad r' = \sqrt{x_1^2 + \cdots + x_7^2}. \quad (\text{A.52})$$

Note that when replacing $H_1(r)$ by $H_1(r')$ we still have a solution, because $H_1(r')$ is still harmonic. However we have gained one translational isometry (and lost some of the spherical symmetry in exchange). We can now use (4.8) to T-dualize over x_8 . The resulting solution takes again the form of a fundamental string solution and we can 'localize' it by making the inverse replacement

$H_1(r') \rightarrow H_1(r)$. Since T-duality relates type IIA to type IIB we have mapped the fundamental IIA/B string to the IIB/A string.

T-duality parallel to the worldvolume relates the fundamental IIA/B string to the IIB/A pp-wave, as discussed above. One can also show that T-duality orthogonal to the world volume maps the IIA pp-wave to the IIB pp-wave and vice versa. Finally parallel T-duality maps a Dp -brane to a $D(p-1)$ -brane and orthogonal T-duality maps a Dp -brane to $D(p+1)$ -brane. In most of these cases T-duality has to be combined with localization and/or delocalization. In the case of Dp -branes one has to know the transformation properties of R-R gauge fields under T-duality, see [14].

Solution XIV : The answers are given in the text.

Solution XV : Using the formula (5.3) for dimensional reduction we get the metric

$$d\bar{s}^2 = -H^{-1}dt^2 + d\bar{x}^2 , \quad (\text{A.53})$$

the Kaluza-Klein gauge field

$$A_t = H^{-1} - 1 \quad (\text{A.54})$$

and the Kaluza-Klein scalar

$$e^{2\sigma} = H , \quad (\text{A.55})$$

where

$$H = K + 1 = 1 + \frac{Q}{r^{D-4}} . \quad (\text{A.56})$$

In D dimensions, Q is the lightlike momentum along the y -direction, whereas from the $(D-1)$ -dimensional point of view Q is the electric charge with respect to the Kaluza-Klein gauge field. Once the y -direction is taken to be compact, $y \simeq y + 2\pi R$, the parameter Q is discrete, $Q = \frac{\tilde{Q}}{R}$.

We have to take K to be independent of y in order to compactify this direction. If $D = 4$, the transverse harmonic function K takes the form $K \sim \log r$ and diverges for $r \rightarrow \infty$. The solution is not asymptotically flat. This is typical for brane-like solutions, where the number of transverse dimensions is smaller than three. Examples are seven-branes in ten dimensions, cosmic strings in four dimensions and black holes in three dimensions.

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