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# **Introduction to Statistical Physics**

An Instructor's Guide

Kerson Huang

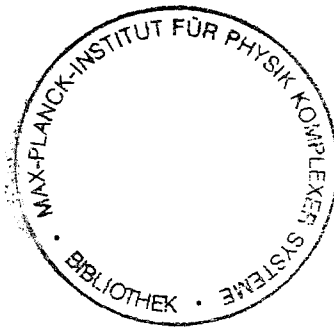
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# Introduction to Statistical Physics

## An Instructor's Guide



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## Chapter 1

### 1.1

Mass of water =  $10^6$  g, heated by  $20^\circ\text{C}$ .

Heat needed  $Q = 2 \times 10^7 \text{ cal} = 8.37 \times 10^7 \text{ J} = 23.2 \text{ kwh}$ .

Work needed =  $mgh = 14 \times 150 \times 29000 = 6.09 \times 10^7 \text{ ft-lb} = 22.9 \text{ kwh}$ .

### 1.2

Work done along various paths are as follows

ab:

$$\int_a^b P dV = NkT_1 \int_a^b \frac{dV}{V} = NkT_1 \ln \frac{V_b}{V_a}$$

cd:

$$P_d(V_d - V_b) = NkT_3 \left(1 - \frac{V_b}{V_d}\right)$$

de:

$$NkT_3 \int_d^e \frac{dV}{V} = NkT_3 \ln \frac{V_e}{V_d}$$

No work is done along  $bc$  and  $ea$ . The total work done is the sum of the above. Heat absorbed equals total work done, since internal energy is unchanged in a closed cycle.

### 1.3

(a)

$$\alpha = \frac{1}{V} \frac{\partial V}{\partial T} = \frac{bV_0 T^{b-1}}{T_0^b V}$$

(b)

$$\Delta V = \frac{bV_0 T^{b-1}}{T_0^b} \Delta T$$

$$P = \frac{NkT}{V} = \frac{NkT_0^b}{V_0} T^{1-b}$$

$$\text{Work done} = P \Delta V = bNk \Delta T$$

### 1.4

Consider an element of the column of gas, of unit cross section, and height between  $z$  and  $z+dz$ . The weight of the element is  $-gdM$ , where  $dM$  is the mass of the element:  $dM = mndz$ , where  $m$  is the molecular mass, and  $n = P/kT$  is the local density, with  $P$  the pressure. For equilibrium, the weight must equal the pressure differential:  $dP = -gdM$ . Thus,  $dP/P = -(mg/kT)dz$ . At constant  $T$ , we have  $dp/P = dn/n$ . Therefore

$$n(z) = n(0)e^{-mgz/kT}$$

1.5

No change in internal energy, and no work is done. Therefore total heat absorbed  $\Delta Q = \Delta Q_1 + \Delta Q_2 = 0$ . That is, heat just pass from one body to the other. Suppose the final temperature is  $T$ . Then

$\Delta Q_1 = C_1(T - T_1)$ ,  $\Delta Q_2 = C_2(T - T_2)$ . Therefore

$$T = \frac{C_1 T_1 + C_2 T_2}{C_1 + C_2}$$

1.6

Work done by the system is  $-\int H dM$ . Thus the work on the system is

$$\int H dM = \frac{\kappa}{T} \int H dH = \frac{\kappa H^2}{2T}$$

1.7

Consider the hysteresis cycle in the sense indicated in Fig.1.6. Solve for the magnetic field:

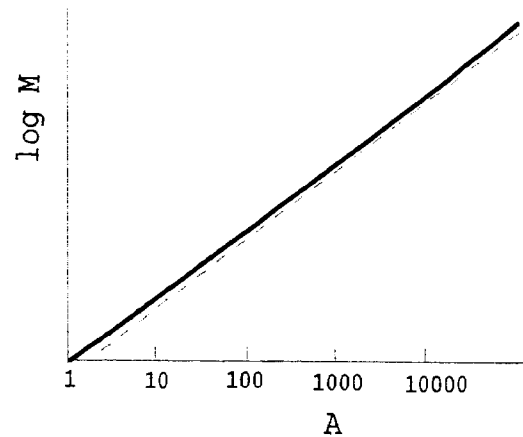
$$H = \pm H_0 + \tanh^{-1}(M/M_0)$$

(+ for lower branch, - for upper branch.). Using  $W = -\int H dM$ , we obtain

$$\begin{aligned} W &= -\int_{-M_0}^{M_0} dM [H_0 + \tanh^{-1}(M/M_0)] - \int_{M_0}^{-M_0} dM [-H_0 + \tanh^{-1}(M/M_0)] \\ &= -4M_0 H_0 \end{aligned}$$

1.8

A log log plot of mass vs.  $A$  is shown in the following graph. The dashed line is a straightline for reference.



## Chapter 2

2.1

Use the  $dQ$  equation with  $P, T$  as independent variables:

$$dQ = C_P dT + [(\partial U/\partial P)_T + P(\partial V/\partial P)_T] dP$$

For an ideal gas  $(\partial U/\partial P)_T = 0$ ,  $P(\partial V/\partial P)_T = -V$ . Thus

$$dQ = C_P dT - V dP.$$

The heat capacity is given by

$$C = C_P - V(\partial P/\partial T)_{\text{path}}.$$

The path is  $P = aV^b$ , or equivalently  $P^{b+1} = a(NkT)^b$  by the equation of state. Hence

$$V(\partial P/\partial T)_{\text{path}} = [ab/(b+1)]V(NkT)^b T^{-1} = bNk/(b+1). \text{ Therefore}$$

$$C = C_P - \frac{b}{b+1} Nk$$

This correctly reduces to  $C_P$  for  $b = 0$ .

2.2

Use a Carnot engine to extracted energy from 1 gram of water between 300 K and 290 K.

Max efficiency  $\eta = 1 - (290/300) = 1/30$ .

$$W = \eta C \Delta T = \frac{1}{30} (4.164 \text{ J g}^{-1} \text{K}^{-1} \times 1 \text{ g} \times 10 \text{ K}) = 1.39 \text{ J}$$

$$\text{Gravitational potential energy} = 1 \text{ g} \times 9.8 \text{ kg s}^{-2} \times 110 \text{ m} = 1.08 \text{ J}$$

2.3

The highest and lowest available temperatures are, 600 F = 588.7 K and 70 F = 294.3 K.

The efficiency of the power plant is  $W/Q_1 = 0.6[1 - (294.3/588.7)] = 0.3$ .

In one second:  $W = 10^6 \text{ J}$ .

So  $Q_2 = 2.33 \times 10^6 \text{ J} = C_V \Delta T$ . Use  $C_V = 4.184 \text{ J g}^{-1} \text{K}^{-1}$ ,

Flow rate =  $6000 \times (0.305 \text{ m})^3$

$$\Delta T = \frac{2.33 \times 10^6 \text{ J}}{(4.184 \text{ J g}^{-1} \text{K}^{-2}) 6000 (0.305 \text{ m})^3 10^6 \text{ cm}^2/\text{m}^3} = 3.27 \times 10^{-3} \text{ K}$$

2.4

(a)

Since water is incompressible, a unit mass input gives a unit mass output.

The net heat supplied per unit mass is  $\Delta Q = C(T_1 - T) - C(T - T_2)$ ,

where  $C$  is the specific heat of water (per unit mass.) In steady state  $v^2/2 = \Delta Q$ . This gives

$$v = \sqrt{2\Delta Q} = \sqrt{2C(T_1 + T_2 - 2T)}$$

(b)

The entropy depends on the temperature like  $\ln T$ . A unit volume of water from each of the input streams has total entropy  $\ln T_1 + \ln T_2$ . This makes two unit volumes in the output stream, with entropy  $2 \ln T$ . Therefore the change in entropy is  $\ln(T^2/T_1 T_2) \geq 0$ . Thus  $T \geq \sqrt{T_1 T_2}$ , and

$$v_{\max} = \sqrt{2C} \left| \sqrt{T_1} - \sqrt{T_2} \right|$$

2.5

(a)

$$PV_1^\gamma = 2P_0 V_0^\gamma, PV_2^\gamma = 2P_0 V_0^\gamma \\ (V_1/V_2)^\gamma = 2. \left[ (\bar{L} + a)/(L - a) \right]^\gamma = 2.$$

$$\frac{a}{\bar{L}} = \frac{2^{1/\gamma} - 1}{2^{1/\gamma} + 1}$$

(b)

$$\Delta U = \Delta Q - W, \Delta Q = 0.$$

$$C_V \Delta T = -W, \Delta T = -W/C_V.$$

$$T_1 = 2T_0 + \Delta T = 2T_0 - (W/C_V), T_2 = T_0 - \Delta T = T_0 + (W/C_V).$$

$$P = \frac{RT_1}{V_1} = \frac{R[2T_0 - (W/C_V)]}{A(L + a)}$$

(c)

$$W = A \int_0^a dx (P_1 - P_2)$$

$$P_1 = 2P_0 V_0^\gamma / [A(L + x)]^\gamma, P_2 = P_0 V_0^\gamma / [A(L - x)]^\gamma.$$

$$W = \frac{P_0 V_0}{\gamma - 1} \left( 1 - \frac{a}{L - \gamma} \right) \left( 1 - \frac{2a}{L} \right)$$

2.6

(a)

$$PV = U/3, U = \sigma VT^4.$$

$$P = \sigma T^4/3.$$

$$dS = dQ/T = (dU + PdV)/T.$$

Integrate along paths with  $T = \text{const}$ ,  $V = \text{const}$ .

$$S = \frac{4}{3} \sigma VT^3.$$

(b)

$S = \text{Constant}$ .  $\therefore T^3 \sim V^{-1}$ . Thus

$$T \sim R^{-1}$$

2.7

The heat absorbed by an ideal gas in an isothermal process is

$$\Delta Q = NkT \ln(V_f/V_i)$$

where  $V_f$  and  $V_i$  are respectively the final and initial volume. The temperature  $T$  in this formula is the ideal-gas temperature.

Draw a Carnot cycle on the  $PV$  diagram, and label the corners 1234 clockwise from the upper left.

The heat absorbed at the upper temperature  $T_2$ , and the heat rejected at the lower temperature  $T_1$ , are

$$Q_2 = NkT_2 \ln(V_2/V_1)$$

$$Q_1 = NkT_1 \ln(V_3/V_4)$$

Because 23 and 12 lie on adiabatic lines, we have

$$V_2 T_2^{\gamma-1} = V_3 T_1^{\gamma-1}$$

$$V_1 T_2^{\gamma-1} = V_4 T_1^{\gamma-1}$$

Dividing one equation by the other yields  $V_2/V_1 = V_3/V_4$ .

The efficiency of the cycle is therefore

$$\eta = 1 - \frac{Q_1}{Q_2} = 1 - \frac{T_1}{T_2}$$

2.8

Diesel cycle:

$$Q_2 = C_P(T_3 - T_2)$$

$$Q_1 = C_V(T_4 - T_1)$$

$$\eta = 1 - (Q_1/Q_2) = 1 - \gamma^{-1}[(T_4 - T_1)/(T_3 - T_2)]$$

We have  $P_3 = P_2$ , hence

$$T_3/T_2 = V_3/V_2 = r_c$$

The processes 12 and 34 are adiabatic, with  $TV^{\gamma-1} = \text{constant}$ .  $V_4 = V_1$ .

Thus

$$T_3 V_3^{\gamma-1} = T_4 V_1^{\gamma-1}$$

$$T_2 V_2^{\gamma-1} = T_1 V_1^{\gamma-1}$$

Using the three relations derived, we obtain

$$\eta = 1 - \frac{1}{\gamma} \frac{r_c^\gamma - 1}{r_c^{\gamma-1}(r_c - 1)}$$

2.9

Otto cycle:

$$Q_2 = C_V(T_3 - T_2)$$

$$Q_1 = C_V(T_4 - T_1)$$

$$\eta = 1 - (Q_1/Q_2) = 1 - [(T_4 - T_1)/(T_3 - T_2)]$$

The processes 12 and 34 are adiabatic, with  $TV^{\gamma-1} = \text{constant}$ . We have

$$V_4 = V_1, V_3 = V_2 \text{ Thus}$$

$$T_1 V_1^{\gamma-1} = T_2 V_2^{\gamma-1}$$

$$T_3 V_2^{\gamma-1} = T_4 V_1^{\gamma-1}$$

Taking the ratio of these equations, we have

$$T_2/T_1 = T_3/T_4 = r^{\gamma-1}$$

Thus

$$\eta = 1 - r^{1-\gamma}$$

2.10

First note  $T_b/T_a = V_b/V_a = 2$ .

|     | Work done                          | Heat absorbed            |
|-----|------------------------------------|--------------------------|
| a→b | $P_a(V_b - V_a) = P_a V_a = NkT_a$ | $C_P \Delta T = C_P T_a$ |
| b→c | 0                                  | $-C_V T_a$               |
| c→a | $-\int P dV = -NkT_a \ln 2$        | $-NkT_a \ln 2$           |

$$W \text{ (Net work done)} = NkT_a(1 - \ln 2)$$

$$Q_2 \text{ (Heat absorbed)} = C_P T_a = \frac{5}{2} NkT_a$$

$$\eta = \frac{W}{Q_2} = \frac{2}{5}(1 - \ln 2) = 0.12$$

In comparison,  $\eta_{\text{Carnot}} = 1 - (T_b/T_a) = 0.5$ .

2.11

First note  $T_2 = 4T_1$ . The  $P, V, T$  for the points  $A, B, C, D$  are as follows:

|     | $P$    | $V$               | $T$    |
|-----|--------|-------------------|--------|
| $A$ | $P_1$  | $V_1 = NkT_1/P_1$ | $T_1$  |
| $B$ | $2P_1$ | $2V_1$            | $4T_1$ |
| $C$ | $2P_1$ | $V_1$             | $2T_1$ |
| $D$ | $P_1$  | $2V_1$            | $2T_1$ |

(a)

Heat supplied along

$$ACB : C_V T_1 + C_P(2T_1) = \left(\frac{3}{2} + 5\right) NkT_1 = \frac{13}{2} NkT_1$$

$$ADB : C_P T_1 + C_V(2T_1) = \left(\frac{5}{2} + 3\right) NkT_1 = \frac{11}{2} NkT_1$$

$$AB : \Delta U + \Delta W = \frac{3}{2} Nk(2T_1) + \frac{3}{2} P_1 V_1 = 6NkT_1$$

(b)

$$\text{Heat capacity} = \Delta Q/\Delta T = 6NkT_1/3T_1 = 2Nk$$

(c)

$$\text{Work done} = P_1 V_1 = NkT_1. \text{ Heat absorbed} = \text{Heat absorbed along } ACB \\ = (13/2)NkT_1.$$

$$\eta = \frac{2}{13}$$

2.12

(a)

Since no work is being done, and the temperatures diverge, heat must be transferred from the colder body to the hotter body, with no other effect, and this violates the Clausius statement of the second law.

(b)

The assertion is not true for physical black bodies, because they cannot be point-like but have finite size. Even if the two bodies have identical shapes, their optical images are not reciprocal. That is, the radiation from one body may form an image that is larger than the other body, and thus not completely absorbed by the other body.

## Chapter 3

### 3.1

(a)

For a adiabatic process  $dS = 0$ , and the  $TdS$  equations give

$$C_V dT = -(\alpha T / \kappa_T) dV$$

$$C_P dT = \alpha TV dP$$

Dividing one by the other, we obtain

$$C_P / C_V = \kappa_T [-V(\partial P / \partial V)_S] = \kappa_T / \kappa_S$$

(b)

$$C_V dT + (\alpha T / \kappa_T) dV = C_P dT - \alpha TV dP. \text{ Put}$$

$$dT = (\partial T / \partial P)_V dP + (\partial T / \partial V)_P dV.$$

Equate the coefficients of  $dP$  and  $dV$  on both sides. One of them gives

$$C_P - C_V = (\alpha TV / \kappa_T)(\partial V / \partial T)_P = \alpha^2 TV / \kappa_T.$$

(c)

Using  $U = A + TS$ ,  $H = G + TS$  (enthalpy), we have

$$\begin{aligned} C_V &= (\partial U / \partial T)_V = (\partial A / \partial T)_V + S + T(\partial S / \partial T)_V = T(\partial S / \partial T)_V \\ &= -T(\partial^2 A / \partial T^2)_V \end{aligned}$$

$$\begin{aligned} C_P &= (\partial H / \partial T)_P = (\partial G / \partial T)_P + S + T(\partial S / \partial T)_P = T(\partial S / \partial T)_P \\ &= -T(\partial^2 G / \partial T^2)_P \end{aligned}$$

### 3.2

The Sack-Tetrode equation is

$$S = Nk[(5/2) - \ln(n\lambda^3)], \text{ where } n = N/V, \text{ and } \lambda = \sqrt{2\pi\hbar^2/mkT}.$$

(a)

$$A = U - TS = (3/2)NkT - TS = NkT \ln(n\lambda^3) - NkT.$$

$$G = A + PV = NkT \ln(n\lambda^3).$$

(b)

Write  $\ln(n\lambda^3) = \ln n + \ln \lambda^3$ . The second term is a function of  $T$  only.

$$\mu = (\partial A / \partial N)_{V,T} = kT \ln(n\lambda^3) + NkT(\partial \ln n / \partial N)_{V,T} - kT = kT \ln(n\lambda^3).$$

$$\mu = (\partial G / \partial N)_{P,T} = kT \ln(n\lambda^3) + NkT(\partial \ln n / \partial N)_{P,T} = kT \ln(n\lambda^3).$$

### 3.3

The force on the bead is  $(P - P_a)A - mg$ , where

$P$  = pressure in gas,

$P_a = 1 \text{ atm}$ .

The equation of motion for the displacement  $x$  is  $m\ddot{x} = (P - P_a)A - mg$ .

In equilibrium the pressure in the gas is  $P_0 = P_a + (mg/A)$ .

The volume is  $V_0 = RT/P_0$ .

Assume adiabatic oscillations:  $PV^\gamma = \text{const}$ .

This implies  $dP = -\gamma(P/V)dV \approx -\gamma(P_0/V_0)Ax$ .

$$P = P_0 + dP \approx P_0 - \gamma(P_0/V_0)Ax.$$

Thus  $m\ddot{x} + (\gamma A^2 P_0^2 / RT)x = 0$ .

The frequency of oscillations is

$$\omega = AP_0 \sqrt{\gamma / RT}$$

### 3.4

Let the equilibrium pressure and temperature be  $P_0, T_0$ . Under an infinitesimal displacement  $x$ , suppose the pressure of compartment 1 changes by  $dP$ . Since the process is adiabatic, we have  $PV^\gamma = \text{constant}$ , or  $(dP/P) + \gamma(dV/V) = 0$ . In terms of the temperature, we have  $TV^{\gamma-1} = \text{constant}$ , or  $(dT/T) + (\gamma - 1)(dV/V) = 0$ .

(a)

For compartment 1, we have to first order

$$\begin{aligned} dP &= -\frac{\gamma P_0 x}{L} \\ dT &= -\frac{(\gamma - 1)T_0 x}{L} \end{aligned}$$

For compartment 2, replace  $x$  by  $-x$ .

(b)

The force acting on the piston is  $dF = AdP$ . The equation of motion for  $x$  is  $dF = M\ddot{x}$ , where  $M$  is the mass of the piston. Thus  $\ddot{x} + (\gamma AP_0 / ML)x = 0$ , and the frequency of small oscillations is

$$\omega = \sqrt{\gamma AP_0 / ML}$$

(c)

Due to the finite thermal conductivity of the piston, heat flows back and forth between the two compartment, because of the oscillation in the temperature difference. Assume that the temperatures change so slowly that at any moment we regard heat conduction as taking place between two heat reservoirs of fixed temperatures. When an amount of heat  $dQ$  flows from 1 to 2, the entropy increase is  $dS = (dQ/T_2) - (dQ/T_1)$ . Thus

$$\frac{dS}{dt} = \left( \frac{1}{T_2} - \frac{1}{T_1} \right) \frac{dQ}{dt} = \frac{K(\Delta T)^2}{T_1 T_2} \approx K \left( \frac{\Delta T}{T_0} \right)^2$$

The temperature difference is

$$(\Delta T)^2 = (T_1 - T_2)^2 = (2dT)^2 = \frac{4(\gamma - 1)^2 T_0^2 x^2}{L^2}$$

Hence

$$\frac{dS}{dt} = ax^2$$

where  $a = 4K(\gamma - 1)^2 / L^2$ .

(d)

Energy dissipation, which has so far been ignored, occurs at the rate  $T_0 dS/dt = aT_0 x^2$ . The time average of this rate is  $\frac{1}{2}aT_0 x_0^2$ , where  $x_0$  is the amplitude of oscillation. The energy of oscillation is  $E = \frac{1}{2}M\omega^2 x_0^2$ . In one period of oscillation, the energy dissipated is  $\Delta E = (2\pi/\omega)\frac{1}{2}aT_0 x_0^2$ . This gives a fractional dissipation per cycle

$$\frac{\Delta E}{E} = \frac{2\pi T_0}{aM\omega^3}$$

3.5

(a)

$$P = -\left(\frac{\partial A}{\partial V}\right)_T = a_0(v_0 - v)$$

(b)

$\kappa_T = -v^{-1}(\partial v/\partial P)_T = (a_0 v)^{-1}$   
 $\alpha = v^{-1}(\partial v/\partial T)_P = -v^{-1}(\partial P/\partial T)_V(\partial v/\partial P)_T$ , by chain rule.

$$\alpha = \frac{1}{a_0 v} \frac{da_0}{dT}$$

(c)

$$\mu = \left(\frac{\partial A}{\partial N}\right)_{V,T} = a_0(v_0^2 - v^2) - f$$

3.6

For this problem it is important to use the entropy expression with arbitrary  $C_V$ , instead of setting it to  $(3/2)k$ . Write the adiabatic condition as

$$\Delta S = \Delta S_1 + \Delta S_2 = 0, \text{ or}$$

$$(N_1 + N_2)k \ln(V_f/V_i) + (N_1 C_{V1} + N_2 C_{V2}) \ln(T_f/T_i) = 0.$$

Thus,  $T_f/T_i = (V_i/V_f)^\zeta$ , where  $\zeta = k(N_1 + N_2)/(N_1 C_{V1} + N_2 C_{V2})$ .

This means  $TV^\zeta = \text{constant}$ . Putting  $T = PV/Nk$ , where  $N = N_1 + N_2$ , we have

$$PV^\xi = \text{constant}$$

where

$$\xi = \zeta + 1 = \frac{N_1(C_{V1} + k) + N_2(C_{V2} + k)}{N_1 C_{V1} + N_2 C_{V2}} = \frac{n_1 C_{P1} + n_2 C_{P2}}{n_1 C_{V1} + n_2 C_{V2}}$$

3.7

(a)

Since the disks are thin, we can assume that their temperatures always remain uniform.

Let the final temperature be  $T$ .

The changes in temperatures are respectively  $\Delta T_1 = T - T_1$ ,  $\Delta T_2 = T - T_2$ .

For simplicity write  $C_{P1} = C_1$ ,  $C_{P2} = C_2$ .

The amounts of heat absorbed are respectively  $\Delta Q_1 = C_1 \Delta T_1$ ,  $\Delta Q_2 = C_2 \Delta T_2$ .

Since the system is isolated  $\Delta Q_1 + \Delta Q_2 = 0$ . This gives

$$T = \frac{C_1 T_1 + C_2 T_2}{C_1 + C_2}$$

(b)

Consider the instant when the two temperatures are  $T'_2, T'_1$ , ( $T'_2 > T'_1$ ).

When an amount of heat  $dQ$  flows from 2 to 1, the entropy increase is  $dS = (dQ/T'_1) - (dQ/T'_2)$ .

We can express  $dQ$  in terms of the  $dT'$  through  $dQ = C_1 dT'_1 = -C_2 dT'_2$ .

Thus we can rewrite  $dS = C_1(dT'_1/T'_1) + C_2(dT'_2/T'_2)$ .

$$\Delta S = C_1 \int_{T_1}^{T'} \frac{dT'_1}{T'_1} + C_2 \int_{T_2}^{T'} \frac{dT'_2}{T'_2} = C_1 \ln \frac{T}{T_1} + C_2 \ln \frac{T}{T_2}$$

3.8

The relations are straightforward mappings from a  $PV$  system to a magnetic system.

3.9

(a)

The desired expression are straightforward mappings of those for a  $PV$  system.

(b)

The first relation is the condition that  $dA$  be an exact differential. The second is obtained by using the equation of state  $M = \kappa H/T$ .

(c)

The chain rule states  $(\partial T/\partial H)_S(\partial H/\partial S)_T(\partial S/\partial T)_H = -1$ .

From (b) we have  $(\partial H/\partial S)_T = -T^2/(\kappa H)$ .

By definition, the heat absorbed at constant  $H$  is given by  $TdS = C_H dT$ .

Thus  $(\partial S/\partial T)_H = C_H/T$ .

3.10

(a)

The important property to verify is that at constant  $T$  the entropy decreases as the magnetic field  $H$  increases.

(b)

Isothermal magnetization:  $dT = 0$ .

The heat absorbed is

$dQ = C_M dT - HdM = -HdM$ . Therefore

$$\Delta Q = -\int_0^H HdM = -\frac{\kappa H^2}{2T_0}$$

(c)

Adiabatic cooling:  $dQ = 0$ .

From  $dQ = C_M dT - H dM$  we obtain

$dT = (H/C_M) dM = (\kappa/aT^2) M dM$ . Multiply both sides by  $T^2$  and integrate:

$$\int_{T_0}^{T_1} T^2 dT = (\kappa/a) \int_M^0 M dM.$$

This gives  $T_1^3 = T_0^3 - (\kappa/2a) M^2$ , or

$$T_1^3 = T_0^3 - \frac{\kappa^3 H^2}{2aT_0^2}$$

This becomes negative when the magnetic field  $H$  is sufficiently large. However, the equation becomes invalid long before that happens, for it is based on Curie's law, which is valid only for weak fields.

## Chapter 4

### 4.1

The system is in contact with a heat reservoir, but initially not in equilibrium with it. Let the stages of the process be labeled A,B,C:.. We first calculate the heat absorbed  $\Delta Q$ , and the entropy change  $\Delta S$  of the system.

(A) Water cools from  $20^\circ\text{C}$  to  $0^\circ\text{C}$ .

$$\Delta Q = C_P \Delta T = -10 \times 4180 \times 20 \text{ J} = -8.36 \times 10^5 \text{ J}.$$

$$\Delta S = \int dQ/T = C_P \int dT/T = C_P \ln(T_f/T_i) = 41800 \ln(273/293) = -2.96 \times 10^3 \text{ J/deg}.$$

(B) Solidification at  $0^\circ\text{C}$ .

$$\Delta Q = -10 \times 3.34 \times 10^5 \text{ J} = -3.34 \times 10^6 \text{ J}.$$

$$\Delta S = \Delta Q/T = -3.34 \times 10^6 / 273 = -1.22 \times 10^4 \text{ J/deg}.$$

(C) Ice cools from  $0^\circ\text{C}$  to  $-10^\circ\text{C}$ .

$$\Delta Q = C_P' \Delta T = -10 \times 2090 \times 10 \text{ J} = -2.09 \times 10^5 \text{ J}.$$

$$\Delta S = C_P' \ln(T_f/T_i) = 20900 \ln(263/273) = -7.80 \times 10^2 \text{ J/deg}.$$

Total heat absorbed by system:  $\Delta Q_{\text{sys}} = -4.39 \times 10^6 \text{ J}$

Total entropy change of system:  $\Delta S_{\text{sys}} = -1.39 \times 10^4 \text{ J/deg}.$

The reservoir has a fixed temperature  $T_0 = -10^\circ\text{C}.$

The total heat absorbed by reservoir equals that rejected by the system:

$$\Delta Q_{\text{res}} = 4.39 \times 10^6 \text{ J}.$$

Entropy change of reservoir:

$$\Delta S_{\text{res}} = \Delta Q_{\text{res}}/T_0 = 4.39 \times 10^6 / 263 = 1.67 \times 10^4 \text{ J/deg}.$$

$$\Delta S_{\text{universe}} = \Delta S_{\text{sys}} + \Delta S_{\text{res}} = 2.8 \times 10^3 \text{ J/deg}$$

### 4.2

Let  $P_0, T_0$  be the pressure and absolute temperature at the triple point. Let  $L$  be the extensive latent heat (not specific latent heat.) Since the solid-gas transition can be made either via a direct path or a solid-liquid-gas path, we must have

$$L_{\text{sublimation}} = L_{\text{melt}} + L_{\text{vap}}$$

Vaporization:  $dP/dT \approx L_{\text{vap}}/TV = PL_{\text{vap}}/NkT^2.$

$$P = P_0 \exp \left[ \frac{L_{\text{vap}}}{NkT_0} \left( 1 - \frac{T_0}{T} \right) \right]$$

Melting:  $dP/dT = L_{\text{vap}}/T\Delta V.$

$$P = P_0 + \frac{L_{\text{melt}}}{\Delta V} \ln \frac{T}{T_0}$$



Sublimation:  $dP/dT \approx P(L_{\text{vap}} + L_{\text{melt}})/NkT^2$ .

$$P = P_0 \exp \left[ \frac{L_{\text{vap}} + L_{\text{melt}}}{NkT_0} \left( 1 - \frac{T_0}{T} \right) \right]$$

4.3

$$dP/dT = \ell/T\Delta v = [1.44 \text{ J}/(18 - 20)\text{cm}^3]T^{-1}.$$

$$\therefore dT/dP = -c_0T, \text{ where } c_0 = 1.39 \text{ cm}^3/\text{J}.$$

4.4

(a)

At a given  $v > v_0$ , the dashed line lies at a lower free energy than the solid line. The latter represents a "stretched" that fills the whole volume. The former represent a liquid drop at specific volume  $v_0$  that does not fill up the entire volume. This is therefore the preferred state of the liquid. At  $v = v_0$  the pressure is zero.

(b)

Now assume that the liquid coexists with its vapor, treated as an ideal gas. We are in the transition region of a first-order phase transition. At the given temperature, the liquid and gas have fixed densities, which must be consistent with the requirement of equal pressure  $P$  and chemical potential  $\mu$ . Denote quantities for the liquid with subscript 1, and those for the vapor with subscript 2:

$$P_1 = a_0(v_0 - v),$$

$$\mu_1 = a_0(v_0^2 - v^2) - f,$$

$$P_2 = nkT,$$

$$\mu_2 = kT \ln(n\lambda^3).$$

where  $P_1, \mu_1$  were obtained in Prob.3.5, and  $\mu_2$  was given in Prob.3.2, with  $\lambda = \sqrt{2\pi\hbar^2/mkT}$ . Thus, the conditions determining  $v$  and  $n$  are

$$a_0(v_0 - v) = nkT$$

$$a_0(v_0^2 - v^2) - f = kT \ln(n\lambda^3)$$

From the first equation, we see that  $v_0 - v > 0$ . It approaches zero as  $nT \rightarrow 0$ .

(c)

Small  $n$  corresponds to  $(v_0 - v) \rightarrow 0$ . The second equation becomes  $-f \approx kT \ln(n\lambda^3)$ . Thus

$$n\lambda^3 \approx \exp(-f/kT)$$

4.5

(a)

$$dP/dT = \ell/[T(v_2 - v_1)] \approx \ell/Tv_2 = \ell/[T(kT/P)]. \text{ Hence}$$

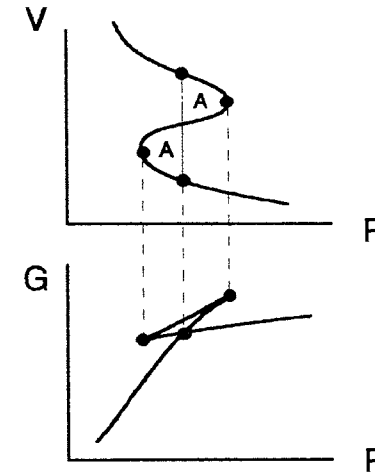
$$\frac{T}{P} \frac{dP}{dT} = \frac{\ell}{kT}$$

(b)

| $T$ (K) | $\ell$ (ergs/g)    |
|---------|--------------------|
| 0.2     | $8.21 \times 10^7$ |
| 0.4     | 9.37               |
| 0.6     | 10.5               |
| 0.8     | 11.8               |
| 1.0     | 13.1               |
| 1.2     | 14.4               |

4.6

The accompanying sketch shows  $G = \int VdP$ . The system skips the closed loop in the graph of  $G$ , because it is higher than need be



4.7

(a)

$$A(V, T) = -RT \ln(V - b) - (a/V) + f(T)$$

$$\text{As } V \rightarrow \infty, A(V, T) \rightarrow -RT \ln V + f(T)$$

This should approach the ideal gas result (Prob.3.2)  $RT[\ln(n\lambda^3) - 1]$ .

Therefore, up to an additive constant,

$$f(T) = -RT \left( 1 + \frac{3}{2} \ln T \right)$$

(b)

$$C_V = -T(d^2 f/dT^2) = (3/2)R, \text{ which is a constant.}$$

4.8

$$TdS = C_V dT + T(\partial P/\partial T)_V dV = 0.$$

$$dT/dV = -(T/C_V)(\partial P/\partial T)_V = -(RT/C_V)(V-b)^{-1}.$$

Integrating this yields

$$\ln T = -(R/C_V) \ln(V-b) + \text{constant}.$$

Thus the adiabatic condition is

$$T(V-b)^{R/C_V} = \text{constant}$$

When  $a = b = 0$ , the system reduces to an ideal gas, for which

$$\frac{R}{C_V} = \frac{C_P - C_V}{C_V} = \gamma - 1$$

Thus we recover  $TV^{\gamma-1} = \text{constant}$ .

4.9

The second virial coefficient for the van der Waal gas is given by  $c_2 = b - (a/RT)$ . A rough fit is

$$\begin{aligned} b &\approx 17 \text{ cm}^3/\text{mole} \\ a &\approx 2100R \text{ deg cm}^3/\text{mole} \end{aligned}$$

4.10

Let  $\Delta V = V_1 - V_2$  be the difference in volume across the transition line. Consider variations along the transition line, such as going from  $a$  to  $b$ , as illustrated in the sketch. The chain rule says

$$\left(\frac{\partial \Delta V}{\partial T}\right)_P \left(\frac{\partial T}{\partial P}\right)_{\Delta V} \left(\frac{\partial P}{\partial \Delta V}\right)_T = -1$$

This gives

$$\left(\frac{\partial P}{\partial T}\right)_{\Delta V} = -\frac{(\partial \Delta V/\partial T)_P}{(\partial \Delta V/\partial P)_T} = \frac{\alpha_1 - \alpha_2}{\kappa_{T1} - \kappa_{T2}}$$

If the transition line refers to a second-order phase transition, then across this line  $\Delta V = 0$ , while the differences in  $\alpha$  and  $\kappa$  are nonzero. Thus

$$\frac{dP}{dT} = \frac{\Delta\alpha}{\Delta\kappa_T}$$

## Chapter 5

5.1

$$n = 2.70 \times 10^{19} \text{ atoms/cm}^3$$

$$v = 2 \times 10^5 \text{ cm/s.}$$

$$N = nv/6 \approx 10^{24} \text{ s}^{-1} \text{ cm}^{-2}.$$

5.2

Let  $tV_0$  be the volume of the room, and  $V$  be the volume under consideration

The probability of finding an atom in  $V$  is  $V/V_0$ .

The probability of finding it elsewhere is  $1 - (V/V_0)$ .

Since there are  $N$  independent atoms, the probability of finding none in  $V$  is

$$p = \left(1 - \frac{V}{V_0}\right)^N = \exp\left(N \ln\left(1 - \frac{V}{V_0}\right)\right)$$

For small  $V/V_0$  we can use the expansion  $\ln(1 - (V/V_0)) \approx -V/V_0$ . Thus

$$p \approx \exp(-NV/V_0)$$

Under STP,

$$N = \frac{V_0}{22.4 \text{ liter mole}^{-1}} \times (6.02 \times 10^{23} \text{ mole}^{-1})$$

For  $V_0 = 27 \times 10^3$  liter, we have

$$\text{for } V = 1 \text{ cm}^3: \quad p \approx \exp(-2.7 \times 10^{19}) \approx 10^{-10^{19}}$$

$$\text{for } V = 1 \text{ A}^3: \quad p \approx \exp(-2.7 \times 10^{-5}) \approx 1 - 2.7 \times 10^{-5} = 0.99997$$

5.3

Let  $n = N/V$

Probability of finding one atom in  $dV = ndV$ .

Probability of finding no atom in  $dV = 1 - ndV$ .

Probability of finding no atom in  $V = \exp(-nV)$ .

$p(r)dr = \text{Prob.}(\text{one atom between } r, r+dr) \times \text{Prob.}(\text{no atom in sphere of radius } r)$

$$p(r) = 4\pi nr^2 \exp\left(-\frac{4}{3}\pi nr^3\right)$$

5.4

For the beam to remain well-collimated, the atoms should suffer no scattering by the air in the chamber along the flight path of length  $L$ . The condition is therefore  $\lambda > L$ , where  $\lambda \approx (n\sigma)^{-1}$  is the mean-free-path, where  $n$  is the density

of the air, and  $\sigma$  is the cross section for a collision between atoms in the beam with an air molecule. Thus

$$n < \frac{1}{L\sigma}$$

For a rough estimate, take  $\sigma \approx 10^{-16} \text{ cm}^2$ . This gives  $n < 10^{-15} \text{ cm}^{-3}$ . The estimate can be refined by using a more precise value for  $\sigma$ .

5.5

(a)

The mass density of water is  $1 \text{ g cm}^{-3}$ . This corresponds to a number density  $n = 2 \times 10^{23} \text{ cm}^{-3}$ . Thus  $\lambda = 5 \times 10^{16} \text{ cm}$ .

(b)

The rate of reaction is  $R = NI\sigma$ , where  $N$  is the number of nucleons,  $I$  is the neutrino flux, and  $\sigma$  is the reaction cross section. A person of mass  $150 \text{ kg}$  contains  $N = 10^{29}$  nucleons. Thus  $R = 5 \times 10^{-10} \text{ s}^{-1}$ .

The collision time is  $\tau = R^{-1} = 2 \times 10^9 \text{ s} \approx 70 \text{ yrs}$ .

Thus, one gets hit by a neutrino about once in a lifetime.

5.6

Following the hint, the answer is obtained straightforwardly:

$$C_n = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2} + 1)} \xrightarrow{n \rightarrow \infty} \frac{n}{2} \ln \pi - \frac{n}{2} \ln \frac{n}{2} + \frac{n}{2}$$

5.7

(a)

From (5.28),  $\Gamma(E, V) = (\partial\Phi/\partial E)\Delta$ , where

$$\Phi = V^N \int_{p_1^2 + \dots + p_n^2 < E} dp_1 \dots dp_n$$

with  $n = 3N$ . Thus

$$\Gamma(E, V) = K_0 V^n \Sigma_n(\sqrt{E}) = K_0 V^n n C_n E^{(n-1)/2}$$

(b) Using  $S = k \ln \Gamma$ , we have, up to an additive constant,

$$\begin{aligned} \frac{S(E)}{Nk} &= \ln V + \frac{\ln C_{3N}}{N} + \ln E^{3/2} + O\left(\frac{\ln N}{N}\right) \\ &= \ln(V E^{3/2}) + O\left(\frac{\ln N}{N}\right) \end{aligned}$$

5.8

(a)

By the same reasoning as in the last problem, we obtain  $\Gamma(E, V) = K_0 \Sigma_n(\sqrt{E})$ , where  $n = 6N$ . There is no volume dependence in the limit  $V \rightarrow \infty$ , because the particles are confined by the harmonic oscillator potential.

(b) Transcribing the result of the last problem, we have

$$\frac{S(E)}{Nk} = \ln E^3 + O\left(\frac{\ln N}{N}\right)$$

5.9

The solution has been outlined in the problem.

5.10

Let the mean-free-path be  $\lambda \approx 10^{-5} \text{ cm}$ . To be away from the origin by a distance  $L$ , a total of  $(L/\lambda)^2$  random steps would have to be taken. Since each step lasts a collision time  $\tau \approx 10^{-10} \text{ s}$ , the total time required is  $\tau(L/\lambda)^2$ . For  $L = 1 \text{ cm}$  the time is: 1 sec. For  $L = 1 \text{ m}$  the time is  $10^4 \text{ sec}$ .

5.11

For one coordinate, the probability of return after  $k$  collisions is  $p = \sqrt{2/\pi k}$ , according to Prob.5. For the  $N$ -particle state to recur, all  $6N$  coordinates have to return at the same time. As an estimate of this probability, assume they all do so after  $k$  collisions, where  $k$  is some fixed number. The probability is then  $p^{6N}$ . Thus

$$\text{Recurrence time} \approx p^{-6N} = \exp\left(6N \ln \frac{1}{p}\right)$$

in units of the collision time. For macroscopic  $N$ , this number is so large that the unit is irrelevant..

## Chapter 6

6.1

Let  $\lambda = (2mkT)^{-1}$ .

$$\begin{aligned}\langle \epsilon \rangle &= \frac{\int d^3p \epsilon f(\mathbf{p})}{\int d^3p f(\mathbf{p})} = \frac{1}{2m} \frac{\int_0^\infty dp p^4 \exp(-\lambda p^2)}{\int_0^\infty dp p^2 \exp(-\lambda p^2)} = \frac{3}{2} kT \\ \langle \epsilon^2 \rangle &= \frac{\int d^3p \epsilon^2 f(\mathbf{p})}{\int d^3p f(\mathbf{p})} = \frac{1}{4m^2} \frac{\int_0^\infty dp p^6 \exp(-\lambda p^2)}{\int_0^\infty dp p^2 \exp(-\lambda p^2)} = \frac{15}{4} (kT)^2 \\ \langle \epsilon^2 \rangle - \langle \epsilon \rangle^2 &= \frac{3}{2} (kT)^2\end{aligned}$$

6.2

The energy distribution is defined through  $P(E)dE = f(p)4\pi p^2 dp$ , where  $f(p)$  is the Maxwell-Boltzmann distribution of momentum. Using  $E = p^2/2m$ , we obtain

$$P(E) = c_0 \sqrt{E} e^{-E/kT}$$

where  $c_0 = n\pi^{-1/2}(kT)^{-3/2}$ .

6.3

The density is obtained by integrating the distribution function over the momentum. The result is

$$n(z) = n(0)e^{-mgz/kT}$$

6.4

Using the equation of state of the ideal gas, we obtain  $P^{(1-\gamma)/\gamma}T = C_0$ . After some manipulation this leads to

$$\frac{dP}{P} = \frac{\gamma}{\gamma-1} \frac{dT}{T} - \frac{mg}{kT} dz = \frac{\gamma}{\gamma-1} \frac{dT}{T}$$

Thus  $T$  changes with height  $z$  according to

$$k \frac{dT}{dz} = -\frac{\gamma-1}{\gamma} mg$$

This can be integrated to yield

$$kT(z) = kT_0 - \frac{\gamma-1}{\gamma} mgz$$

For  $T_0 = 300$  K and  $\gamma = 7/5$ , the temperatures becomes zero at  $z = 3.17 \times 10^4$  m.

(b)

From the above, we find

$$\frac{dP}{P} = -\frac{mg}{kT} dz$$

Using the expression for from the last part, we can integrate this to obtain

$$\frac{P}{P_0} = \left(1 - \frac{\gamma-1}{\gamma} \frac{mgz}{kT_0}\right)^{\gamma/(\gamma-1)}$$

6.5

There is an effective temperature-dependent potential  $U(x)$ , given through  $\exp(-U/kT) = c_0(1 + \gamma x)$ .

6.6

The answer is

$$\frac{n_1(r)}{n_2(r)} = \exp[\omega^2 r^2 (m_1 - m_2)/2kT]$$

6.7

(a)

The most probable velocity is that at the maximum of the speed distribution. This will be obtained in (c).

(b)

The pressure is given by

$$P = \int_{p_x > 0} d^3p 2p_x v_x f(\mathbf{p}) = \frac{1}{3} \int d^3p \frac{p^2}{\sqrt{p^2 + m^2}} f(\mathbf{p})$$

where we have used  $v_x = p_x/\sqrt{p^2 + m^2}$ . Write

$$\frac{p^2}{\sqrt{p^2 + m^2}} = \sqrt{p^2 + m^2} - \frac{m^2}{\sqrt{p^2 + m^2}}$$

The second can be neglected in the ultra-relativistic limit  $p^2 \gg m^2$ . Comparing  $P$  with the energy density  $U/V = \int d^3p \sqrt{p^2 + m^2} f(\mathbf{p})$ , we obtain

$$PV \rightarrow \frac{1}{3}U$$

(c)

The velocity distribution  $f(v)$  is defined by

$$f(v)d^3v = C \exp(-\beta m v / \sqrt{1-v^2}) d^3p$$

Using  $p = mv/\sqrt{1-v^2}$ , we obtain

$$f(v) = \frac{Cm^3}{(1-v^2)^{5/2}} \exp\left(-\frac{m}{kT\sqrt{1-v^2}}\right)$$

The non-relativistic limit corresponds to  $m(1 - v^2)^{-1/2} \approx \frac{1}{2}mv^2 + m + O(v^2)$

Now return to part (a). The most probable velocity  $v_0$  corresponds to the maximum of the speed distribution  $4\pi v^2 f(v)$ . It is given by the root of the equation

$$(1 - v^2)^{3/2} + \frac{5}{2}v^2(1 - v^2)^{1/2} - \frac{m}{2kT}v^2 = 0$$

The non-relativistic and ultra-relativistic limits are (with  $c$  restored)

$$\begin{aligned} \frac{v_0}{c} &\approx \sqrt{\frac{2kT}{mc^2}} \quad (kT \ll mc^2) \\ \frac{v_0}{c} &\approx 1 - \left(\frac{mc^2}{5kT}\right)^2 \quad (kT \gg mc^2) \end{aligned}$$

(d)

Relativistic effects become noticeable when  $kT/mc^2$  is appreciable, say, at 10%. For  $H_2$  this corresponds to  $kT = 0.1 \times 2 \text{ GeV}$ , or  $T = 2 \times 10^{12} \text{ K}$ .

6.8

(a)

The distribution is proportional to the velocity distribution  $\exp(-mv_x^2/2kT)$ . Substitute  $v_x = c(f - f_0)/f_0$  and then normalize the distribution. The result is

$$P(f) = \left(\frac{2\pi kT f_0}{mc^2}\right)^{-1/2} \exp\left(-\frac{mc^2}{2kT f_0^2}(f - f_0)^2\right)$$

(b)

The variance is

$$\overline{(f - f_0)^2} = \frac{\int_0^\infty df (f - f_0)^2 \exp(-\lambda(f - f_0)^2)}{\int_0^\infty df \exp(-\lambda(f - f_0)^2)}$$

where  $\lambda = mc^2/(2kT f_0^2)$ . Change the variable of integration to  $\nu = \sqrt{\lambda}(f - f_0)$ . The lower limit of integration becomes  $-\sqrt{\lambda}f_0 = -\sqrt{mc^2/2kT}$ . This can be replaced by  $-\infty$  when  $kT \ll mc^2$ , which is true in usual laboratory conditions. We then obtain

$$\overline{(f - f_0)^2} = \frac{kT}{mc^2} f_0^2$$

(c)

The line width is given by the square root of the variance, and thus inversely proportional to  $\sqrt{m}$ . The  $H_2$  line width is therefore broader than that of  $O_2$  by a factor  $\sqrt{32/2} = 4$ .

6.9

(a)

$$f(p) \propto e^{-\beta cp},$$

$$U/N = c\bar{p} = 3kT,$$

$$C_V = 3Nk.$$

(b)

$$PV = \frac{1}{3}U = NkT.$$

6.10

Follow the hints and directions given in the problem.

6.11

$$\begin{aligned} W &= \int_{v_x > v_0} d^3p v_x f(\mathbf{p}) = C \int_{mv_0}^\infty dp_x v_x e^{-\lambda p_x^2} \left[ \int_{-\infty}^\infty dp_y \exp(-\lambda p_y^2) \right]^2 \\ &= n \sqrt{\frac{kT}{2\pi m}} \exp\left(-\frac{v_0^2}{2mkT}\right) \end{aligned}$$

6.12

(a) The escape velocity is  $v_c = \sqrt{2GM/R} \approx 10^4 \text{ m/s}$ . This is to be compared with the most probable speed at STP  $v_0 = \sqrt{2kT/m} \approx 2.2 \times 10^3 \text{ m/s}$ . The fraction of gas that can escape is

$$f = \frac{C}{n} \int_{mv_c}^\infty dp 4\pi p^2 e^{-p^2/2mkT} = \frac{4}{\sqrt{\pi}} \int_y^\infty dx x^2 e^{-x^2}$$

where  $C = n(2\pi mkT)^{-3/2}$  and  $y = v_c/v_0$ . Using the results of Prob.6.10(b), we obtain

$$f \approx \frac{2y}{\sqrt{\pi}} e^{-y^2}$$

With  $y \approx 4.5$ , we find  $f \approx 5 \times 10^{-8}$ .

(b)

The time it takes for an atom to go from sea level to the top of the atmosphere through random collisions is

$$t \approx \frac{L^2}{\lambda v_c} = 3 \times 10^{11} \text{ s} \approx 10^4 \text{ yr}$$

where  $L$  = height of atmosphere  $\approx 100 \text{ km}$ ,  $\lambda$  = mean-free-path  $\approx 3 \times 10^{-7} \text{ m}$ .

6.13

(a)

The number of atoms with momentum magnitude between  $p$  and  $p + dp$  is  $V 4\pi p^2 f(p) dp$ , where  $V$  is the volume, and  $f(p)$  is the Maxwell-Boltzmann distribution. Thus

$$\Delta N = 4\pi V \int_{p_0}^{\infty} dp p^2 f(p)$$

$$\Delta E = 4\pi V \int_{p_0}^{\infty} dp p^2 \frac{p^2}{2m} f(p)$$

Using the results of Prob 6.10(b), and  $N = nV$ , and  $E = \frac{3}{2}NkT$ , we obtain the fractional changes

$$\frac{\Delta N}{N} = 2\pi y^{1/2} e^{-y}$$

$$\frac{\Delta E}{E} = \frac{4\pi}{3} y^{3/2} e^{-y}$$

for  $y = \epsilon_0/kT \gg 1$ .

(b)  
From  $kT = \frac{2}{3}E/N$ , we obtain  $k\Delta T = \frac{2}{3}[(\Delta E/N) - E(\Delta N/N^2)]$ , hence

$$\frac{\Delta T}{T} = \frac{\Delta E}{E} - \frac{\Delta N}{N} \approx 2\pi y^{1/2} \left( \frac{2}{3}y - 1 \right) e^{-y}$$

Taking the logarithm of the equation for  $\Delta N/N$ , we have  $\ln(N/\Delta N) = y - \ln(2\pi y^{1/2})$ , which gives

$$y \approx \ln(N/\Delta N) + \ln(2\pi \ln(N/\Delta N)^{1/2}) = \ln(\pi N/\Delta N) + \ln \ln(N/\Delta N)$$

We then find, to leading order,

$$\frac{\Delta T}{T} \approx \frac{4}{3} \frac{\Delta N}{N} \sqrt{\ln \frac{\Delta N}{N}}$$

6.14

(a)

Let the axis along the needle be labeled 1, and a perpendicular axis 2. The moments of inertia about these axes are  $I_1$ ,  $I_2$ , with  $I_2 \gg I_1$ . By the equipartition of energy we have

$$\frac{J_1^2}{2I_1} = \frac{J_2^2}{2I_2} = \frac{kT}{2}$$

where  $J_i$  is the components of angular momentum along the axis  $i$ . Thus

$$\frac{J_2}{J_1} = \sqrt{\frac{I_2}{I_1}} \gg 1$$

That is, the angular momentum is nearly parallel to the axis of the needle.

(b)

The equipartition of energy states

$$\frac{1}{2}C\overline{V^2} = \frac{1}{2}kT$$

This gives  $\sqrt{\overline{V^2}} = 6.5 \mu V$ .

6.15

(a)

Take 1 mole of  $N_2$ . The mass is 28 g.

For  $v = 7 \text{ km/s}$ , the kinetic energy is

$$\text{K.E.} = \frac{1}{2}Mv^2 = 0.5(0.028)(7000)^2 = 686 \text{ kJ.}$$

When this energy is converted into heat, the temperature rise is of the order of

$$\Delta T = \text{K.E.}/k = 5500 \text{ K.}$$

Thus, the astronauts would be fried.

(b)

A constant deceleration  $a$  is equivalent to the application of a potential  $max$ , where  $x$  is the distance, and  $m$  is the mass of an air molecule. The Boltzmann factor gives a relative density distribution

$$\frac{\Delta n}{n} = \exp\left(-\frac{max}{kT}\right)$$

which equals the fractional change in pressure  $\Delta P/P$  at constant temperature. The difference in pressure between the points  $x = 0$  and  $x = L$  is therefore  $\Delta P = P_0[1 - \exp(-mL/kT)]$ .

(c)

Let the total mass of air be  $M = Na$ , where  $N$  is the total number of air molecules. The force is

$$F = A\Delta P = \frac{AP_0maL}{kT} = \frac{P_0V}{NkT}Nma = Nma = Ma$$

The stopping time is  $t = v/a$ , which corresponds to a distance  $\frac{1}{2}at^2 = v^2/2a$ . The work done is therefore

$$W = \frac{Fv^2}{2a} = \frac{1}{2}Mv^2$$

Thus the translational kinetic energy is completely converted to mechanical work, and  $\Delta T = 0$ .

(d) The translational velocity of the air must be, at all times, much smaller than sound velocity, relative to the walls of the container.

## Chapter 7

7.1

Particle flux:  $I_N = \int_{v_x > 0} d^3p v_x f(\mathbf{p})$

Energy flux:  $I_E = \int_{v_x > 0} d^3p v_x f(\mathbf{p}) m v^2 / 2$

Average energy of an escaped particle

$$\frac{I_E}{I_N} = \frac{m \int_0^\infty dv v^5 \exp(-mv^2/2kT)}{2 \int_0^\infty dv v^3 \exp(-mv^2/2kT)} = 2kT$$

Thus, the escaped atoms come to thermal equilibrium at a temperature  $T_1$ , with  $\frac{3}{2}kT_1 = 2kT$ . Hence  $T_1 = \frac{4}{3}T$ . This assumes that the total amount of gas escaped is so small that the temperature of the original system is unchanged.

7.2

Let  $n_1, n_2$  denote the densities of U-238 and U-235 respectively. Flux  $= n\sqrt{kT/2\pi m}$ .

After one stage of effusion,

$$\frac{n_1}{n_2} = \left(\frac{n_1}{n_2}\right)_0 \sqrt{\frac{m_2}{m_1}}$$

After  $k$  stages,

$$\left(\frac{n_1}{n_2}\right)_k = \left(\frac{n_1}{n_2}\right)_0 \left(\frac{m_2}{m_1}\right)^{k/2}$$

Find  $k$  such that  $n_1 = n_2$ .

$$k = \frac{2 \ln(n_1/n_2)_0}{\ln(m_1/m_2)} = \frac{2 \ln(99.27/0.75)}{\ln(238/235)} = 775$$

7.3

In an adiabatic process  $PV^\gamma = \text{constant}$ . Using the equation of state, we find that  $P(NkT/P)^\gamma = \text{constant}$ , or  $P^{1-\gamma}T^\gamma = \text{constant}$ . Differentiating this relation with respect to  $P$  we obtain

$$\left(\frac{\partial T}{\partial P}\right)_S = \frac{\gamma - 1}{\gamma} \frac{T}{P}$$

The particle density is  $n = P/kT$ . Thus

$$\left(\frac{\partial n}{\partial P}\right)_S = \frac{1}{kT\gamma}$$

Hence

$$\kappa_S = \frac{1}{n k T \gamma}$$

$$c = \sqrt{\frac{kT\gamma}{m}}$$

7.4

Test the condition  $K/c \ll C_P$ . Using the data given, we find  $K/c = 1.44 \times 10^{-5}$ ,  $C_P = 0.24T$ , in the mixed unit given. Thus the condition is well-fulfilled, and shows that sound propagated adiabatically.

7.5

From (7.19)  $\partial^2 \rho / \partial t^2 + \rho \nabla \cdot \partial \mathbf{u} / \partial t = 0$ . Instead of the Euler equation  $\rho \partial \mathbf{u} / \partial t = -\nabla P$ , use the Navier-Stokes equation (7.48). Then in first-order approximation (7.21) is replaced by

$$\frac{\partial^2 \rho}{\partial t^2} - \nabla^2 P + \frac{4\nu}{3} \nabla^2 (\nabla \cdot \mathbf{u}) = 0$$

where  $\nu$  is the viscosity. Use the continuity equation  $\rho \nabla \cdot \mathbf{u} = -\partial \rho / \partial t$ , and convert  $\nabla^2 P$  to  $\nabla^2 \rho$  as in (7.22). The result is

$$\frac{\partial^2 \rho}{\partial t^2} - \frac{1}{c^2} \nabla^2 \rho - \frac{4\nu}{3\rho} \nabla^2 \frac{\partial \rho}{\partial t} = 0$$

For a sinusoidal wave  $\rho = \rho_0 + \rho_1 \exp(ikx - i\omega t)$ , the last term is  $i(4\nu k\omega/3\rho_0)\rho_1$ . Thus the damping coefficient is  $4\nu k\omega/3\rho_0$ .

7.6

The one-dimensional diffusion equation has solution

$$n(x, t) = \frac{N}{\sqrt{4\pi Dt}} \exp\left(-\frac{x^2}{4\pi Dt}\right)$$

The gas is characterized by the diffusion constant  $D$ . Suppose the detector has spatial resolution  $\Delta x$ . We want to find the time  $t$  at which  $n(L, t)\Delta x = 1$ . That leads to the implicit equation

$$t = \frac{L^2}{4\pi D} \frac{1}{\ln(N\Delta x/\sqrt{4\pi Dt})}$$

In the first approximation, we put  $t = L^2/4\pi D$  on the right side. This gives

$$t = \frac{L^2}{4\pi D} \frac{1}{\ln(N\Delta x/L)}$$

The logarithm is not very sensitive to  $\Delta x$ .

7.7

The insulating power  $\eta$  is the inverse of the coefficient of thermal conductivity. Thus  $\eta \propto \sigma\sqrt{m}$ , where  $\sigma$  is the collision cross section, and  $m$  the mass of the molecule of the gas. Assuming that the molecular diameter increases like  $m^{1/3}$  we have

$$\eta \propto m^{5/6}$$

To double  $\eta$ , we need to increase  $m$  by a factor  $2^{6/5} = 2.3$ . To double the insulating power of air, we would need a gas of molecular weight 69.

7.8

This is a hypothetical exercise, since we are ignoring an important heat source, the radiation from the sun. (See Prob.10.3).

The total rate of heat generated is  $\frac{4}{3}\pi R^3 W$ , and this must equal the rate of heat radiated  $4\pi R^2 \sigma T_1^4$ , where  $T_1$  is the surface temperature, and  $\sigma$  is Stefan's constant. This gives the surface temperature

$$T_1 = \left( \frac{RW}{3\sigma} \right)^{1/4}$$

In the interior, the rate of heat generation per unit volume is  $\rho W$ , and this equals  $\nabla \cdot \mathbf{q}$ , where  $\mathbf{q}$  is the heat flux vector. Using  $\mathbf{q} = -\kappa \nabla T$ , we have the equation for the temperature distribution  $\nabla^2 T = -\rho W / \kappa$ . Assuming that  $T$  is spherically symmetric, and using spherical coordinates, we obtain

$$\frac{d}{dr} r^2 \frac{dT(r)}{dr} = -\frac{\rho W}{\kappa} r^2$$

Integration of this equation, observing that  $T$  cannot be singular at  $r = 0$ , gives

$$T(r) = T_0 - \frac{\rho W}{6} r^2$$

where  $T_0$  is the temperature at  $r = 0$ . The surface temperature is

$$T_1 = T_0 - \frac{\rho W}{6} R^2$$

Thus

$$T_0 = \frac{1}{6} \rho W R^2 + \left( \frac{RW}{3\sigma} \right)^{1/4}$$

7.9

(a)

The heat absorbed by per unit volume is  $dQ = -\nabla \cdot \mathbf{q} dt$ , which defines the heat flux vector  $\mathbf{q}$ . Putting  $dQ = T ds$ , we have

$$\frac{\partial s}{\partial t} + \frac{1}{T} \nabla \cdot \mathbf{q} = 0$$

(b)

Consider the heat flux due to heat conduction  $\mathbf{q} = -\kappa \nabla T$ . Write  $(\nabla \cdot \mathbf{q})/T = \nabla \cdot (\mathbf{q}/T) - \mathbf{q} \cdot \nabla(1/T)$ . The last equation can be rewritten

$$\frac{\partial s}{\partial t} = -\nabla \cdot \frac{\mathbf{q}}{T} + \kappa \left( \frac{\nabla T}{T} \right)^2$$

The second term, which is always positive, is the rate of irreversible entropy production.

7.10

Suppose the thickness of the ice sheet is  $x$ . Consider a unit square of the ice sheet. The mass of the sheet increases at the rate  $\rho dx/dt$ , and generates heat at the rate  $\ell \rho dx/dt$ . This must equal the heat flux, which we can represent as

$$q_x = \frac{\kappa \Delta T}{x}$$

for a small thickness  $x$ . Thus

$$\frac{dx}{dt} = \frac{\kappa \Delta T}{\ell \rho x}$$

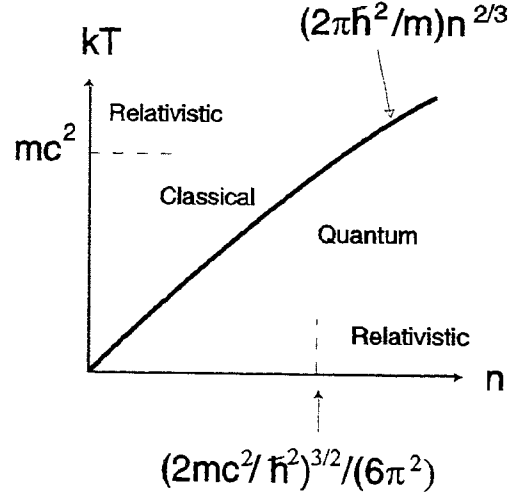


## Chapter 8

8.1

The relativistic energy is  $E = \sqrt{(pc)^2 + (mc^2)^2}$ . In the ultra-relativistic domain we can neglect the mass term and thus  $E = pc$ . The deBroglie wavelength is  $h/p = hc/E$ . The thermal wavelength is therefore proportional to  $hc/kT$ .

8.2



8.3

$$U = \frac{3}{2}PV \approx \frac{3}{2}NkT \left(1 \pm 2^{-5/2}n\lambda^3\right)$$

$$C_V = \frac{\partial U}{\partial T} \approx \frac{3}{2}Nk \left(1 \mp 2^{-7/2}n\lambda^3\right)$$

The upper sign is for fermions, lower sign is for bosons.

8.4

(a)

$$N = \sum_{\lambda} n_{\lambda} = z \sum_{\lambda} \exp(-\beta \epsilon_{\lambda}) = zQ$$

(b)

The internal energy per particle is defined by

$$\frac{U}{N} = \frac{\sum_{\lambda} \epsilon_{\lambda} \exp(-\beta \epsilon_{\lambda})}{\sum_{\lambda} \exp(-\beta \epsilon_{\lambda})}$$

Thus,

$$-\frac{\partial \ln Q}{\partial \beta} = \frac{1}{Q} \sum_{\lambda} \epsilon_{\lambda} \exp(-\beta \epsilon_{\lambda}) = U$$

(c)

$$Q = \sum_{\alpha\beta\gamma} \exp(-\beta(\epsilon_{\alpha}^{\text{trans}} + \epsilon_{\beta}^{\text{rot}} + \epsilon_{\gamma}^{\text{vib}})) = Q_{\text{trans}} Q_{\text{rot}} Q_{\text{vib}}$$

$$\frac{U}{N} = -\frac{\partial}{\partial \beta} (\ln Q_{\text{trans}} + \ln Q_{\text{rot}} + \ln Q_{\text{vib}})$$

$$c_V = \frac{1}{N} \frac{\partial U}{\partial T} = c_{\text{trans}} + c_{\text{rot}} + c_{\text{vib}}$$

8.5

$$Q_{\text{trans}} = V(2\pi)^{-3} 4\pi \int_0^{\infty} dk k^2 \exp(-\beta \hbar^2 k^2 / 2m) = V/\lambda^3.$$

$$U/N = -\partial \ln \lambda^{-3/2} / \partial \beta = \frac{3}{2} kT$$

$$\frac{c_{\text{trans}}}{k} = \frac{3}{2}$$

8.6

(a)

$$\ln Q_{\text{rot}} \approx \ln(1 + \exp(-\beta \hbar^2 / I)) \approx \exp(-\beta \hbar^2 / I)$$

$$\frac{U}{N} = -\frac{\partial \ln Q_{\text{rot}}}{\partial \beta} = \frac{\hbar^2}{I} \exp\left(-\frac{\beta \hbar^2}{I}\right)$$

$$\frac{c_{\text{rot}}}{k} \approx 3 \left(\frac{\beta \hbar^2}{I}\right)^2 \exp\left(-\frac{\beta \hbar^2}{I}\right)$$

(b)

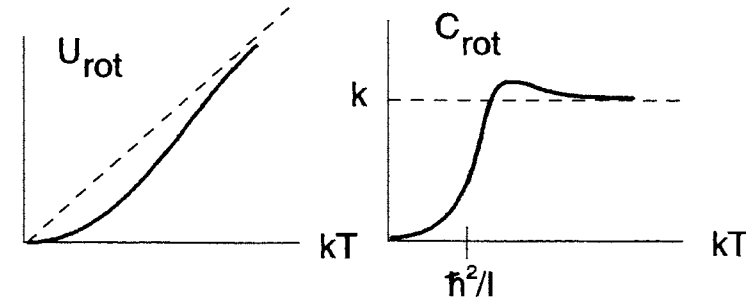
$$Q_{\text{rot}} \approx \int_0^{\infty} d\ell 2\ell \exp(-\beta \hbar^2 \ell^2 / 2I) \propto \beta^{-1}$$

$$U/N \approx kT$$

$$\frac{c_{\text{rot}}}{k} \approx 1$$

(c)

The internal energy rises exponentially from  $T = 0$  to approach a linear behavior. The qualitative behaviors are as shown in the accompanying sketch.



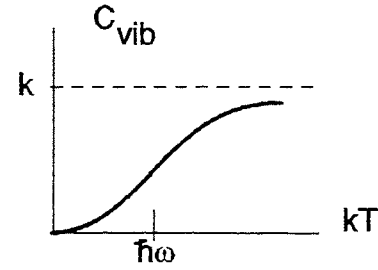
8.7

(a)

$$Q_{\text{vib}} = \sum_{n=0}^{\infty} \exp(-\beta \hbar \omega (n + 1/2)) = e^{\lambda/2} (e^{\lambda} - 1)^{-1}$$

$$\frac{U}{N} = -\frac{\partial \ln Q_{\text{vib}}}{\partial \beta} = \frac{\hbar \omega}{2} \frac{e^{\lambda} + 1}{e^{\lambda} - 1} \quad (\lambda = \beta \hbar \omega)$$

$$\frac{c_{\text{vib}}}{k} = e^{-\beta \hbar \omega} \left( \frac{\beta \hbar \omega}{1 - e^{-\beta \hbar \omega}} \right)^2$$



(b)

$$\left\langle n + \frac{1}{2} \right\rangle = -\frac{\partial}{\partial \lambda} \ln Q_{\text{vib}} = \frac{1}{2} \frac{e^{\lambda} + 1}{e^{\lambda} - 1}$$

$$\left\langle \left( n + \frac{1}{2} \right)^2 \right\rangle - \left\langle n + \frac{1}{2} \right\rangle^2 = \frac{\partial^2}{\partial \lambda^2} \ln Q_{\text{vib}} = \frac{e^{\lambda}}{(e^{\lambda} - 1)^2}$$

where  $\lambda = \beta \hbar \omega$ .

8.8

$$kT_{\text{vib}} \approx \hbar \omega$$

$$kT_{\text{rot}} \approx \frac{\hbar^2}{I}$$

For  $\text{H}_2$ ,  $T_{\text{vib}} = 6100 \text{ K}$ ,  $T_{\text{rot}} = 85.4 \text{ K}$ .

From  $T = 0$ , the specific heat rises to  $3k/2$  before it reaches  $T_{\text{rot}}$ , then increases by  $k$  around  $T = T_{\text{rot}}$ , and increases by  $k$  again around  $T = T_{\text{vib}}$ .

8.9

$$\epsilon_n = \gamma_n + b\sigma_n$$

$$\gamma_n = \hbar \omega \left( n + \frac{1}{2} \right)$$

$$\sigma_n = \hbar \omega \left( n + \frac{1}{2} \right)^2$$

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$$\langle \epsilon \rangle = \frac{\sum_n (\gamma_n + b\sigma_n) e^{-\beta(\gamma_n + b\sigma_n)}}{\sum_n e^{-\beta(\gamma_n + b\sigma_n)}}$$

Expanding this to first order in  $b$ , we have

$$\frac{\langle \epsilon \rangle}{\hbar \omega} \approx \frac{\bar{\epsilon}}{\hbar \omega} + b\bar{\nu}^2 - b\beta \hbar \omega (\bar{\nu}^3 - \bar{\nu}^2 \bar{\nu})$$

where  $\nu = \bar{n} + 1/2$ , and a bar denotes average with respect to the unperturbed system with  $b = 0$ .

$$-\frac{\partial}{\partial \lambda} \bar{\nu}^2 = \bar{\nu}^3 - \bar{\nu}^2 \bar{\nu}$$

From Prob.8.7(b) we have

$$\bar{\nu}^2 = \frac{e^{2\lambda} + 6e^{\lambda} + 1}{4(e^{\lambda} - 1)^2}$$

from which we obtain

$$\bar{\nu}^3 - \bar{\nu}^2 \bar{\nu} = -\frac{\partial}{\partial \lambda} \bar{\nu}^2 = \frac{e^{\lambda} (3e^{\lambda} + 4)}{2(e^{\lambda} - 1)^3}$$

where  $\lambda = \beta \hbar \omega$ . Thus

$$\frac{\langle \epsilon \rangle - \bar{\epsilon}}{b\hbar \omega} \approx \frac{e^{2\lambda} + 6e^{\lambda} + 1}{4(e^{\lambda} - 1)^2} - \frac{\lambda e^{\lambda} (3e^{\lambda} + 4)}{2(e^{\lambda} - 1)^3}$$

The specific heat is obtained by differentiating the above with respect to  $T$ .

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## Chapter 9

9.1

The fraction of electrons that can be excited is of the order of  $kT/\epsilon_F$ . Hence the effective density is  $nkT/\epsilon_F$ , where  $n$  is the electron density. The mean-free-path is where  $\sigma$  is the collision cross section.

$$\lambda \approx \frac{\epsilon_F}{n\sigma kT}$$

9.2

$$n = 4.35 \times 10^{27} \text{ cm}^{-3}$$

$$\epsilon_F = 24.6 \text{ MeV}$$

$$\text{Av. energy per nucleon} = \frac{3}{5}\epsilon_F = 14.8 \text{ MeV}$$

9.3

(a)

The Fermi wave number  $k_F$  is given through  $(2s+1)V(4\pi/3)k_F^3 = N$ . Thus,  $k_F = [3n/4\pi(2s+1)]^{1/3}$ .

$$\begin{aligned} p_F &= \hbar k_F \\ \epsilon_F &= \sqrt{p_F^2 c^2 + m^2 c^4} \end{aligned}$$

(b)

$$\begin{aligned} U &= 2V \int_p \left[ \sqrt{(cp)^2 + (mc^2)^2} - mc^2 \right] \\ P &= 2 \int_p p_x v_x = \frac{2}{3} \int_p (\mathbf{p} \cdot \mathbf{v}) \end{aligned}$$

where  $\int_p = \int_{|\mathbf{p}| < p_F} d^3p/h^3$ .

(c)

For  $n^{1/3} \ll mc/\hbar$ , particles near the Fermi surface are non-relativistic:

$$\sqrt{(cp)^2 + (mc^2)^2} \approx mc^2 + \frac{p^2}{2m}$$

We have  $\mathbf{v} = \mathbf{p}/m$ . Hence

$$\begin{aligned} U &\approx V \int_p \frac{p^2}{2m} \\ PV &\approx \frac{2}{3} V \int_p \frac{p^2}{m} = \frac{2}{3} U \end{aligned}$$

For  $n^{1/3} \gg mc/\hbar$ , particles near Fermi surface are ultra-relativistic:

$$\sqrt{(cp)^2 + (mc^2)^2} \approx cp$$

We have  $\mathbf{v} = c\mathbf{p}/p$ . Hence

$$\begin{aligned} U &\approx 2V \int_p cp \\ PV &\approx \frac{2}{3} V \int_p cp = \frac{1}{3} U \end{aligned}$$

(d)

$$\epsilon_F = 6 \times 10^{-5} \text{ eV.}$$

9.4

(a)

Let  $p_{\pm}$  be the Fermi momenta of the spin-up and spin-down gases. The energy of an atom of up(down) spin is

$$\epsilon(H) = \frac{p_{\pm}^2}{2m} \mp \mu H$$

Thus

$$N_{\pm} = \frac{V}{h^3} \frac{4\pi}{3} p_{\pm}^3 = \frac{4\pi V}{3h^3} (2m)^{3/2} [\epsilon(H) \pm \mu H]^{3/2}$$

(b)

For complete polarization, we have  $N_- = 0$ , hence  $\epsilon(H) = \mu H$ , and

$$N_+ = \frac{4\pi V}{3h^3} (4m\mu H)^{3/2}$$

The total density is now  $n = N_+/V$ . The minimum field is

$$H_{\min} = \left( \frac{3\pi^2}{4} \right)^{2/3} \frac{\hbar^2 n^{2/3}}{\mu m}$$

9.5

(a)

Consider a shell of thickness  $dr$  in the gas. Let the pressure differential be  $dP$ . The inward force acting on a patch of the shell of area  $dA$  is  $-dAdP$ . In hydrostatic equilibrium this must equal the gravitational attraction due to the mass at the center. Thus

$$\begin{aligned} -PdA &= \gamma M \rho(r) r^{-2} dA dr \\ \frac{dP}{dr} &= -\frac{\gamma M \rho(r)}{r^2} \end{aligned}$$

(b)

$$P = \frac{2}{5} n \epsilon_F \propto n^{5/3} \propto \rho^{5/3}$$

Thus

$$\frac{d\rho}{\rho^{1/3}} = -K \frac{dr}{r^2}$$

Assuming  $\rho(\infty) = 0$ , we have

$$\rho(r) = \frac{C_0}{r^{3/2}}$$

9.6

(a)

$$N_b = \frac{N}{z^{-1}e^{-\beta\epsilon} + 1}$$

$$N_f = \sum_k \frac{i}{z^{-1}e^{\beta\epsilon_k} + 1}$$

where  $\epsilon_k = \hbar^2 k^2 / 2m$ .

(b)

The condition is  $N_b + N_f = N$ , or

$$\frac{1}{z^{-1}e^{-\beta\epsilon} + 1} + \frac{1}{n\lambda^3} f_{3/2}(z) = 1$$

where  $\lambda = \sqrt{2\pi\hbar^2/mkT}$ .

(c)

For small  $z$ , the condition becomes

$$ze^{\beta\epsilon} + z/(n\lambda^3) = 1$$

Thus

$$z = n\lambda^3 (1 - n\lambda^3 e^{\beta\epsilon})$$

This is valid for  $n\lambda^3 \ll 1$ .

(d)

For high temperatures  $\beta \rightarrow 0$ . Thus  $n\lambda^3 \ll 1$ , and  $z \ll 1$ .From  $N_f = (V/\lambda^3) f_{3/2}(z)$ , or  $n_f \lambda^3 = f_{3/2}(z)$ , we obtain

$$n_f \lambda^3 \approx z = n\lambda^3 (1 - n\lambda^3 e^{\beta\epsilon})$$

$$\frac{n_f}{n} \approx 1 - n\lambda^3 e^{\beta\epsilon}$$

For low temperatures we expect most particles to be in one of the bound states, and thus  $n_f/n \rightarrow 0$ . For  $\beta \rightarrow \infty$ , the condition for  $z$  becomes

$$zf_{3/2}(z) = n\lambda^3 e^{-\beta\epsilon}$$

This means that  $z$  is small, so the condition reduces to  $z^2 = n\lambda^3 e^{-\beta\epsilon}$ . Thus

$$\frac{n_f}{n} \approx \frac{1}{\sqrt{n\lambda^3}} e^{-\beta\epsilon/2}$$

9.7

The probability of finding an electron with energy  $\Delta$  above the Fermi level is  $P(\Delta) = (e^{\beta\Delta} + 1)^{-1}$ . The probability for finding an electron with energy  $\Delta$  below the Fermi level is  $P(-\Delta)$ . Therefore

$$Q(\Delta) = 1 - P(-\Delta) = \frac{1}{e^{\beta\Delta} + 1}$$

9.8

(a)

The number of states in a volume element in momentum space is  $2L^2 dp_x dp_y / h^2$ . The density of states is

$$D(p) = \frac{2A}{(2\pi\hbar)^2}$$

(b)

Obtain  $D(\epsilon)$  through  $D(\epsilon)d\epsilon = D(p)\pi p dp$ :

$$D(\epsilon) = \frac{mA}{2\pi\hbar^2}$$

(c)

$$N = 2 [A/(2\pi)^2] \pi k_F^2$$

$$k_F = \sqrt{\frac{2\pi N}{A}}$$

$$\epsilon_F = \frac{\pi\hbar^2 N}{mA}$$

(d)

$$E = \int_0^{\epsilon_F} d\epsilon \epsilon D(\epsilon) = \frac{\pi\hbar^2}{2m} \frac{N^2}{A}$$

(e)

$$\sigma = \frac{\partial E}{\partial A} = -\frac{\pi\hbar^2}{2m} \left(\frac{N}{A}\right)^2$$

(f)

In 3D  $D(\epsilon) \propto \sqrt{\epsilon}$ , while in 2D  $D(\epsilon)$  is independent of  $\epsilon$ . When the temperature increases from  $T = 0$ , the average energy increases like  $\epsilon_0 + kT$ , and hence the density of states in 3D increases like  $kT$ , whereas it remains constant in 2D. Thus we expect the chemical potential to be less sensitive to temperature in the 2D case. That is, the temperature dependence is weaker.

## Chapter 10

10.1

(a)

The operators  $p$  and  $q$  are hermitian, and defined by  $[p, q] = -i\hbar$ . Solving for  $a$  and  $a^\dagger$ , we have

$$\begin{aligned} a &= (2\hbar m\omega)^{-1/2} p - i(m\omega/2\hbar)^{1/2} q \\ a^\dagger &= (2\hbar m\omega)^{-1/2} p + i(m\omega/2\hbar)^{1/2} q \end{aligned}$$

which give

$$[a, a^\dagger] = 1$$

We can write

$$\begin{aligned} \frac{p^2}{2m} &= \frac{1}{4}\hbar\omega [a^\dagger a + aa^\dagger + (a^2 + a^{\dagger 2})] \\ \frac{1}{2}m\omega^2 q^2 &= \frac{1}{4}\hbar\omega [a^\dagger a + aa^\dagger - (a^2 + a^{\dagger 2})] \end{aligned}$$

Hence

$$\begin{aligned} H &= \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2 = \frac{1}{2}\hbar\omega (a^\dagger a + aa^\dagger) \\ &= \hbar\omega \left( a^\dagger a + \frac{1}{2} \right) \end{aligned}$$

(b)

Define the eigenstate  $|n\rangle$  by

$$\begin{aligned} a^\dagger a |n\rangle &= n |n\rangle \\ \langle n | n \rangle &= 1 \end{aligned}$$

The number  $\langle n | a^\dagger a | n \rangle$  is the norm of the state vector  $a | n \rangle$ , and therefore non-negative.

This means that the eigenvalues of  $a^\dagger a$  cannot be negative.

Multiplying both sides of the first relation by  $a$ , we have

$$aa^\dagger a | n \rangle = na | n \rangle$$

Using  $aa^\dagger = a^\dagger a + 1$ , we have

$$\begin{aligned} (a^\dagger a + 1)a | n \rangle &= na | n \rangle \\ (a^\dagger a)a | n \rangle &= (n-1)a | n \rangle \end{aligned}$$

This means that  $a | n \rangle$  has eigenvalue  $n-1$ .

If  $n$  were not an integer, then successive application of  $a$  would eventually make  $n$  negative.

Since this cannot happen,  $n$  must be an integer.

(c)

We have shown that  $a | n \rangle$  has eigenvalue  $n-1$ .

By the same method we can show  $a^\dagger | n \rangle$  has eigenvalue  $n+1$ . Thus

$$a | n \rangle = C | n-1 \rangle$$

where  $C$  is a constant. The norm of this state is  $C^*C$ , which should be  $n$ . We can choose the real solution

$$C = \sqrt{n}$$

Similarly we can show

$$a^\dagger | n \rangle = \sqrt{n+1} | n+1 \rangle$$

10.2

(a)

The star is completely enclosed by the shell of dust cloud, which absorbs all the radiation from the star. The dust cloud has two surfaces, an outer one and an inner one, and we assume that they have approximately the same area.

Let the temperature of the star be  $T$ , and that of the dust cloud  $T'$ .

Let the power radiated by the star be  $R$ , and that of the dust cloud be  $R'$  from each surface.

The net power outflow from the star is  $R - R'$ . The net influx is  $R'$ . In equilibrium  $R - R' = R'$ . Hence the power radiated to the outside world is

$$R' = \frac{1}{2}R$$

(b)

Since  $R \propto T^4$ ,  $R' \propto T'^4$ , we have  $T'/T = (R'/R)^{1/4} = 2^{-1/4}$ .

10.3

We are ignoring heat generated due to radioactivity in the Earth's interior. (See Prob. 7.8.)

Let the subscripts  $S$  and  $E$  identify quantities relating respectively to the Sun and Earth, which are separated by a distance  $L$ . The radiation per unit surface area is  $\sigma T^4$ . From the viewpoint of the Sun, the fractional solid angle subtended by the Earth is  $\pi R_E^2 / (4\pi L^2)$ . Thus the power received by the Earth is

$$\sigma T_S^4 (4\pi R_S^2) \frac{\pi R_E^2}{4\pi L^2}$$

This must equal the power radiated by the Earth  $\sigma T_E^4 (4\pi R_E^2)$ . Thus the radius of the Earth drops out and we obtain

$$\frac{T_E}{T_S} = \sqrt{\frac{R_S}{2L}}$$

10.4

Energy flows into the house due to transmission of radiation originating outside, and reflection of radiation originating inside. Thus the rate of inflow is  $(1-r)T_0^4 + rT^4$ . The loss rate is  $(1-r)T^4$ . In equilibrium we must have  $(1-r)T_0^4 + rT^4 = (1-r)T^4$ . Thus

$$\frac{T}{T_0} = \left( \frac{1-r}{1-2r} \right)^{1/4}$$

10.5

From (8.61)  $S = T^{-1}(U + PV - N\mu)$ .  
For photons  $\mu = 0$ ,  $PV = U/3$ . Thus

$$S = \frac{4U}{3T} = \frac{4}{3}V\sigma T^3$$

10.6

(a)

From (10.5) the photon density is  $n = \kappa(kT/\hbar c)^3 \approx 4000 \text{ cm}^{-3}$  for  $T = 2.73 \text{ K}$ .

(b)

From Prob. 10.5, the entropy density at  $T = 2.73 \text{ K}$  is  $S/V = k(4\pi/45)(kT/\hbar c)^3$ . The entropy per photon is independent of the temperature:

$$\frac{S}{N} = \left( \frac{4\pi}{45\kappa} \right) k = 1.21k$$

where  $k$  is Boltzmann's constant.

(c)

When  $S = \text{constant}$ ,  $T \propto V^{-1/3}$ . When the radius of the universe doubles, the temperature drops by a factor  $2^{-1/3} = 0.793$ .

10.7

(a)

There are  $3N$  modes, each with energy  $\hbar\omega_0$ .

$$U = \frac{3N\hbar\omega_0}{\exp(\beta\hbar\omega_0) - 1}$$

(b)

$$C_V = \partial U / \partial T \rightarrow 3N(\hbar\omega_0)^2 kT^{-2} \exp(-\beta\hbar\omega_0)$$

(c)

$$A = 3NkT \ln(1 - e^{-\beta\hbar\omega_0})$$

10.8

(a)

$$A = N_{\text{solid}} [-\epsilon + 3kT \ln(1 - e^{-\beta\hbar\omega})] + N_{\text{gas}} kT [1 - \ln(n\lambda^3)]$$

(b)

The chemical potentials must be equal:

$$\ln(n\lambda^3) = -\epsilon + 3kT \ln(1 - e^{-\beta\hbar\omega})$$

(c)

$$P(T) = nkT$$

10.9

(a)

The free energy is

$$A = \phi + 3NkT \ln(1 - e^{-\beta\hbar\omega_0})$$

The condition for equilibrium is  $P = -\partial A / \partial V = 0$ , which gives

$$\frac{\partial \phi}{\partial V} = \frac{3\gamma n \hbar \omega_0}{\exp(\beta \hbar \omega_0) - 1}$$

where  $n = N/V$

(b)

$$V = V_0 + \frac{3\gamma n \hbar \omega_0}{K [\exp(\beta \hbar \omega_0) - 1]}$$

$$\alpha = \frac{3k\gamma n \hbar \omega_0}{KT^2 [\exp(\beta \hbar \omega_0) - 1]^2}$$

where  $n_0 = N/V_0$ .

10.10

(a)

The heat capacity of an electron gas is

$$\frac{C_V}{Nk} \approx \frac{\pi^2}{2} \frac{T}{T_F}$$

$$kT_F = \frac{\hbar^2}{2m} (3\pi^2 n)^{2/3}$$

Thus

$$a = \frac{\pi^2}{2T_F}$$

(b)

The Debye heat capacity for  $T \ll T_D$  is given by

$$\frac{C_V}{Nk} \approx \frac{12\pi^4}{5} \left( \frac{T}{T_D} \right)^3$$

Thus

$$b = \frac{12\pi^4}{5T_D^3}$$

(c)

The plot is  $C_V/nR$  vs.  $T^2$ . Thus, the intercept at  $T^2 = 0$  gives  $Ra$ , and the slope of the line gives  $Rb$ . From the plot, a rough reading gives

$$\begin{aligned} \text{Intercept} &= Ra = 1.2 \times 10^{-3} \text{ cal mole}^{-1} \text{K}^{-2} \\ \text{Slope} &= Rb = 4 \times 10^{-6} \text{ cal mole}^{-1} \text{K}^{-4} \end{aligned}$$

Thus

$$\begin{aligned} a &= 6 \times 10^{-3} \text{ K}^{-1} \\ b &= 2 \times 10^{-6} \text{ K}^{-3} \end{aligned}$$

which lead to

$$\begin{aligned} T_F &= 820 \text{ K} \\ T_D &= 490 \text{ K} \end{aligned}$$

The electron density is  $n \approx 10^{20} \text{ cm}^{-3}$ .

10.11

The energy is  $\epsilon = ak^{2/3}$ , where  $a = \hbar\sqrt{\sigma/\rho}$ .

$$\frac{U}{A} = \int \frac{d^2k}{(2\pi)^2} \frac{\epsilon}{\exp(\beta\epsilon) - 1}$$

Changing the variable of integration to  $x = \beta\epsilon = \beta ak^{2/3}$ , we find

$$\frac{U}{A} = \frac{C_0}{3\pi\hbar^{4/3}} \left( \frac{\rho}{\sigma} \right)^{2/3} (kT)^{7/3}$$

where

$$C_0 = \int_0^\infty dx \frac{x^{4/3}}{e^x - 1} = 1.68$$

## Chapter 11

11.1

In the transition  $P = kT\lambda^{-3}g_{5/2}(1)$ . Since this is proportional to  $T^{5/2}$ , we have  $dP/dT = \frac{5}{2}k\lambda^{-3}g_{5/2}(1)$ . The specific volume of the gas phase at transition is given by its inverse density  $v_0(T) = \lambda^3 [g_{3/2}(1)]^{-1}$ . This represents the change in specific volume in the first-order transition, since the other phase has  $v_0 = 0$ . Thus

$$\frac{dP}{dT} = \frac{5k}{2} \frac{g_{5/2}(1)}{g_{3/2}(1)} \frac{1}{v_0(T)}$$

We can read off the change in specific entropy:

$$s_0 = \frac{5k}{2} \frac{g_{5/2}(1)}{g_{3/2}(1)}$$

The latent heat of transition is  $\ell = Ts_0$ .

11.2

Just above the transition  $z$  is slightly less than 1. Using the expansion given in the problem we have

$$n\lambda^3 = g_{3/2}(z) = g_{3/2}(1) - b\sqrt{\nu} + a\nu$$

where  $\nu = -\ln z$ . Thus the equation for the fugacity is

$$a\nu - b\sqrt{\nu} + c = 0$$

where

$$\begin{aligned} c &= g_{3/2}(1) - n\lambda^3 = g_{3/2}(1) \left[ 1 - \left( \frac{T_c}{T} \right)^{3/2} \right] \\ &\approx \frac{3}{2} g_{3/2}(1) \frac{T - T_c}{T_c} \end{aligned}$$

Choose the solution that goes to zero when  $c \rightarrow 0$ . To order  $c^2$  we have

$$\sqrt{\nu} = c + \left( \frac{a}{b} \right) c^2$$

This leads to the expansion

$$z = 1 - c^2 - \frac{2a}{b} c^3 + \dots$$

11.3

The parametric equation of state is

$$\begin{aligned}\frac{\lambda^3 P}{kT} &= z + \frac{z^2}{2^{5/2}} + \frac{z^3}{3^{5/2}} + \dots \\ \lambda^3 n &= z + \frac{z^2}{2^{3/2}} + \frac{z^3}{3^{3/2}} + \dots\end{aligned}$$

We want to obtain  $P$  as a power series in  $\lambda^3 n$ . To do this, invert the second equation by writing  $z = \lambda^3 n + k_2 (\lambda^3 n)^2 + k_3 (\lambda^3 n)^3 + \dots$ . The coefficients  $k_\ell$  are determined by the second equation. Substitute this into the first equation to obtain an expansion of  $P$ .

The details are as follows: Let  $x = \lambda^3 n$ . We have

$$x = (x + k_2 x^2 + k_3 x^3) + 2^{-3/2}(x + 2k_2 x^2) + 3^{-3/2}x^3 + O(x^4)$$

which gives  $k_2 = 2^{-3/2}$ ,  $k_3 = \frac{1}{4} - 3^{-3/2}$ . Thus

$$\begin{aligned}\frac{\lambda^3 P}{kT} &= (x + k_2 x^2 + k_3 x^3) + 2^{-5/2}(x + 2k_2 x^2) + 3^{-5/2}x^3 + O(x^4) \\ &= x - 2^{-5/2}x^2 + \left(\frac{1}{8} - \frac{2}{9\sqrt{3}}\right)x^3 + O(x^4)\end{aligned}$$

which gives

$$\begin{aligned}a_2 &= -2^{-5/2} \\ a_3 &= \frac{1}{8} - \frac{2}{9\sqrt{3}}\end{aligned}$$

11.4

Putting  $\lambda^3 = cT^{-3/2}$ , we have

$$\begin{aligned}U &= \frac{3}{2}PV = \frac{3}{2c}Vk \begin{cases} T^{5/2}g_{5/2}(z) & (T > T_c) \\ T^{5/2}g_{5/2}(1) & (T < T_c) \end{cases} \\ C_V &= \frac{\partial U}{\partial T} = \frac{3}{2c}Vk \begin{cases} \frac{5}{2}T^{3/2}g_{5/2}(z) + T^{5/2}g_{3/2}(z)\frac{1}{z}\frac{dz}{dT} & (T > T_c) \\ \frac{5}{2}T^{3/2}g_{5/2}(1) & (T < T_c) \end{cases}\end{aligned}$$

We differentiate the relation  $n\lambda^3 = g_{3/2}(z)$  to obtain

$$\frac{1}{z} \frac{dz}{dT} = -\frac{3}{2} \frac{n\lambda^3}{Tg_{1/2}(z)}$$

which approaches zero when  $z \rightarrow 1$ , because  $g_{1/2}(z) \rightarrow \infty$ . Thus  $C_V$  is continuous at  $z = 1$ .

We use  $g_{1/2}(z) = z(d/dz)g_{3/2}(z) = -dg_{3/2}(z)/d\nu$ , where  $\nu = -\ln z$ . From Prob.11.2, we have the expansion near  $z = 1$ :

$$g_{1/2}(z) = b_0\nu^{-1/2} + b_1 + \dots$$

where  $b_0 = 1.7275$ ,  $b_1 = 1.460$ . Thus near  $z = 1$ , or  $\nu = 0$ , we have

$$\frac{1}{z} \frac{dz}{dT} = -\frac{d\nu}{dT} \approx \frac{3}{2} \frac{nc}{b_0} T^{-5/2} \sqrt{\nu}$$

Differentiate  $C_V$ , and setting  $\nu = 0$ , we obtain

$$C'_+ - C'_- = \frac{3}{2} NkT_c \left[ \frac{d}{dT} \left( \frac{1}{z} \frac{dz}{dT} \right) \right]_{\nu=0} = -\xi \frac{Nk}{T_c}$$

where  $+$  and  $-$  indicate approaching the limit from above and below, respectively, and

$$\xi = \frac{27}{16} \left[ \frac{g_{3/2}(1)}{b_0} \right]^2 = \frac{27}{16} \left( \frac{2.612}{1.7275} \right)^2 = 3.86$$

11.5

(a)

Let  $\epsilon = \hbar^2 k^2 / 2m$ . The total number of particles is

$$\begin{aligned}N &= \frac{1}{z^{-1}e^{-\beta\Delta} - 1} + \sum_{k \neq 0} \frac{1}{z^{-1}e^{\beta\epsilon} - 1} \\ \frac{N}{V} &= \frac{1}{V} \frac{1}{z^{-1}e^{-\beta\Delta} - 1} + \frac{1}{\lambda^3} g_{3/2}(z)\end{aligned}$$

The first term is the condensate density. When  $V \rightarrow \infty$ , it is negligible except when  $z = e^{-\beta\Delta}$ .

(b)

The unperturbed transition temperature  $T_0$  is determined by the condition  $n\lambda_0^3 = g_{3/2}(1)$ . The perturbed transition temperature  $T_c$  is determined by the condition  $n\lambda_c^3 = g_{3/2}(e^{-\beta_c\Delta})$ . We can use the approximation

$$g_{3/2}(e^{-\beta_c\Delta}) \approx g_{3/2}(1) - a\sqrt{\Delta/kT_0}$$

where  $a = 3.455$ . Let  $T_c = T_0 + \delta T$ . To lowest order in  $\Delta/kT_0$  we obtain

$$\begin{aligned}\frac{\delta T}{T_0} &= c_0 \sqrt{\frac{\Delta}{kT_0}} \\ c_0 &= \frac{2}{3} \frac{a}{g_{3/2}(1)} = 0.881\end{aligned}$$

11.6

(a)

The internal energy due to phonons is

$$U = \frac{4\pi V}{(2\pi)^3} \int_0^\infty dk k^2 \frac{\hbar ck}{\exp(\beta \hbar ck) - 1} = \frac{\pi^2}{30} \frac{V(kT)^4}{(\hbar c)^3}$$



The heat capacity per unit mass is given by

$$C_{\text{phonon}} = \frac{1}{mN} \frac{\partial U}{\partial T} = \frac{2\pi^2}{15} \frac{k^4}{\rho(\hbar c)^3} T^3$$

where  $\rho$  is the mass density. Using data for liquid helium

$$\begin{aligned} c &= 2.39 \times 10^4 \text{ cm s}^{-1} \\ \rho &= 0.144 \text{ g cm}^{-3} \end{aligned}$$

we have

$$C_{\text{phonon}} = 0.021 T^3 \text{ J g}^{-1} \text{ K}^{-1}$$

where  $T$  is the absolute temperature in K.

(b)

The internal energy per unit volume due to rotons is

$$\frac{U}{V} = \frac{4\pi}{(2\pi)^3} \int_0^\infty dk k^2 \frac{\Delta + \hbar^2(k - k_0)^2/2\sigma}{\exp(\beta\Delta) \exp(\beta\hbar^2(k - k_0)^2/2\sigma) - 1}$$

We are interested in temperatures below 1 K. Since  $\Delta \approx 9$  K, we can ignore the term  $-1$  in the denominator. The main contributions will come from the neighborhood of the roton minimum, i.e., at  $k = k_0$ . Thus we only need to keep the term  $\Delta$  in the numerator. Now change variable of integration to  $q = k - k_0$ . The lower limit of can be replaced by  $-\infty$  for low temperatures. Thus

$$\frac{U}{V} \approx \frac{1}{2\pi^2} \Delta e^{-\beta\Delta} \int_{-\infty}^\infty dq (q + k_0)^2 \exp(-\beta\hbar^2 q^2/2\sigma)$$

We can replace  $(q + k_0)^2$  by  $k_0^2$ . Thus we obtain

$$\begin{aligned} \frac{U}{V} &\approx \frac{k_0^2 \Delta}{\pi} \sqrt{\frac{\sigma k T}{2\pi \hbar^2}} e^{-\Delta/kT} \\ C_{\text{roton}} &\approx \frac{k_0^2 \Delta^2}{\pi \rho} \sqrt{\frac{\sigma k T}{2\pi \hbar^2}} \frac{e^{-\Delta/kT}}{kT^2} \end{aligned}$$

Using data for liquid helium

$$\begin{aligned} \Delta/k &= 8.65 \text{ K} \\ k_0 &= 1.92 \times 10^8 \text{ cm}^{-1} \\ \sigma &= 1.07 \times 10^{-24} \text{ g} \end{aligned}$$

we obtain

$$C_{\text{roton}} \approx \frac{387}{T^{3/2}} \exp\left(-\frac{8.65}{T}\right) \text{ J g}^{-1} \text{ K}^{-1}$$

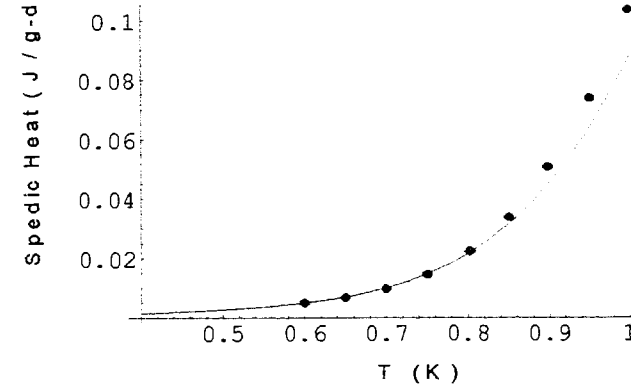
where  $T$  is the absolute temperature in K.

11.7

The specific heat is

$$C = 0.021 T^3 + \frac{387}{T^{3/2}} \exp\left(-\frac{8.65}{T}\right) \text{ J g}^{-1} \text{ K}^{-1}$$

where  $T$  is the absolute temperature in K. The numbers in this formula are based on neutron scattering data independent of the specific heat. The following plot compares this formula to data on specific heat shown as dots. It is an absolute comparison with no adjustable parameters. At  $T = 1$  K, there is already noticeable discrepancy, the main source of which probably comes the approximations we used in the roton specific heat.



11.8

(a)

$$P_1 = -\partial E_0 / \partial V = -(3/5) N \partial \epsilon_F / \partial V = c_1 (\hbar^2/m) n_1^{5/2},$$

$$c_1 = (6\pi^2)^{2/3} / 5.$$

(b)

$$P_2 = kT \lambda^{-3} g_{5/2}(1) = c_2 (m_2/\hbar)^{3/2} (kT)^{5/2},$$

$$c_2 = (2\pi)^{3/2} g_{5/2}(1).$$

(c)

$$kT \ll \epsilon_F,$$

$$kT \ll (\hbar^2/2m) (3n_1/4\pi)^{2/3}.$$

(d)

$$\frac{m_1}{m_2} = \left(\frac{c_2}{c_1}\right)^{2/3} \left(\frac{3}{4\pi}\right)^{10/9} \left(\frac{kT}{\epsilon_F}\right)^{5/3}$$

(e)

For Fermi degeneracy  $m_1/m_2 \ll 1$ . For Bose degeneracy  $T < T_c$ . To fulfill both conditions we must have

$$K \left(\frac{n_1}{n_2}\right)^{5/3} < \frac{m_1}{m_2} \ll 1$$

where  $K$  is a numerical constant.

11.9

In 2D the fugacity  $z$  is determined by

$$N = A \int \frac{d^2k}{(2\pi)^2} \frac{1}{z^{-1} \exp(\beta \hbar k^2 / 2m) - 1}$$

where  $A$  is the area of the system. By expanding the right side in a power series in  $z$  we obtain

$$\frac{N}{A} = \frac{mkT}{2\pi\hbar^2} \ln \frac{1}{1-z}$$

For  $0 \leq z \leq 1$ , there is no upper bound to  $N$ , except when  $T = 0$ . This means that, the states of nonzero momentum can accommodate any  $N$ , and there is no Bose-Einstein condensation. The except occurs at  $T = 0$ , when all particles go into the zero-momentum state.

11.10

For a gas of  $N$  photons with number conservation, the fugacity  $z$  is determined by

$$N = 2V \int \frac{d^3k}{(2\pi)^3} \frac{1}{z^{-1} \exp(\beta \hbar ck) - 1}$$

The zero-momentum state is ignored in the continuum approximation used. By expanding the right side in a power series in  $z$  we obtain

$$n = \frac{2}{\pi^2} \left( \frac{kT}{\hbar c} \right)^3 g_3(z)$$

where  $n = N/V$ , and  $g_n(z) = \sum_{\ell=1}^{\infty} \ell^{-n}$ . The function  $g_3(z)$  is finite at  $z = 1$ , but has an infinite second derivative. We can see by noting that  $z \frac{d}{dz} z \frac{d}{dz} g_3(z) = g_1(z)$  diverges at  $z = 1$ . Thus  $n$  is finite at  $z = 1$ , but cannot be continued beyond  $z = 1$ . There is Bose-Einstein condensation to the zero-momentum state when  $n$  exceeds the critical density

$$n_c = \frac{2}{\pi^2} \left( \frac{kT}{\hbar c} \right)^3 g_3(1)$$

At constant density  $n$  the condensation occurs below the critical temperature

$$T_c = \frac{\hbar c}{k} \left[ \frac{\pi^2 n}{2g_3(1)} \right]^{1/3}$$

## Chapter 12

12.1

(a)

The number of ways to choose the  $n$  atoms to remove from  $N$  sites is

$$N! / [n! (N-n)!]$$

(b)

The number of ways to choose the  $n$  interstitials out of  $M$  is

$$M! / [n! (M-n)!]$$

(c)

The total energy is  $E = n\Delta$ . The phase space volume is

$$\Gamma(n) = \frac{N!M!}{n! (N-n)! n! (M-n)!}$$

Using the Stirling approximation, we obtain the entropy

$$\frac{S}{k} = \ln \Gamma(n) = n \ln \frac{N}{n} - (N-n) \ln \left( 1 - \frac{n}{N} \right) + n \ln \frac{M}{n} - (M-n) \ln \left( 1 - \frac{n}{M} \right)$$

The temperature is defined through

$$\frac{1}{kT} = \frac{1}{k} \frac{\partial S}{\partial E} = \frac{1}{\Delta} \frac{\partial \ln \Gamma(n)}{\partial n}$$

This gives

$$\frac{\Delta}{kT} = \frac{\partial}{\partial n} \ln \Gamma(n) = \ln \left( \frac{N}{n} - 1 \right) + \ln \left( \frac{M}{n} - 1 \right)$$

(d)

The previous equation can be rewritten as

$$\frac{n^2}{(N-n)(M-n)} = \exp \left( -\frac{\Delta}{kT} \right)$$

The low- and high-temperature limits are

$$\begin{aligned} n &\approx \sqrt{NM} \exp(-\Delta/2kT) & (kT \ll \Delta) \\ \frac{1}{n} &\approx \frac{1}{N} + \frac{1}{M} & (kT \gg \Delta) \end{aligned}$$

(e)

$$\frac{n}{N} \approx \exp(-\Delta/2kT)$$

For  $T = 300$  K:  $n/N \approx e^{-20} = 2 \times 10^{-9}$ .

For  $T = 1000$  K:  $n/N \approx e^{-6} = 2.5 \times 10^{-3}$ .

12.2

(a)

Since each link can be pointed left or right independently, the number of ways to choose  $N_+$  links to point right is  $\Gamma = N!/[N_+!(N - N_+)!]$ . We must have  $N_- = N - N_+$ . The entropy is  $S = k \ln \Gamma$ , which leads to

$$\frac{S}{Nk} = -r \ln r - (1 - r) \ln(1 - r)$$

where  $r = N_+/N$  is the fraction of right-pointing links.

(b)

The internal energy is independent of  $N_+$ , and we can set it to zero. Thus the free energy is  $A = -TS$ , where  $T$  is just a constant scale factor.

(c)

The tension  $\tau$  can be obtained from  $dU = 0 = TdS + \tau dL$ , where  $L$  is the length of the chain:

$$L = a(N_+ - N_-) = a(2N_+ - N) = aN(2r - 1)$$

We obtain

$$\frac{\tau}{kT} = -\frac{1}{2aNk} \frac{\partial S}{\partial r} = \frac{1}{2a} \ln \frac{1}{r(1-r)}$$

where  $kT$  is just a scale factor. The tension is never zero. It is minimum when  $r = 1/2$ , and goes to infinity when the chain is fully stretched to the right ( $r = 1$ ) or to the left ( $r = 0$ ).

In this model, "temperature" is not a relevant concept, since energy is irrelevant. The factor  $T$  in  $TdS$  is an arbitrary scale factor.

If we give each left-pointing link an energy  $\epsilon$ , then the total energy would be  $E = \epsilon N_- = \epsilon N(1 - r)$ . The temperature would be given by  $T^{-1} = -\ln[r(1 - r)]$ , apart from a scale factor

12.3

(a)

Assume that a link can be up or down independently. The partition function is the product of the partition functions of the individual links. The possible energies are 0 and  $mga$ . Thus  $Q_N = [1 + \exp(-\beta mga)]^N$ . We have ignored the fact that the energy of the  $n$ th link depends on its height, and therefore on the states of the preceding links. We have also ignored the restriction that the links cannot go above the ceiling.

(b)

$$U = -\frac{\partial \ln Q_N}{\partial \beta} = \frac{Nmga}{\exp(\beta mga) + 1}$$

The length of the chain is  $L = (N - N')a$ , where  $N' = U/(mga)$  is the number of up links. Thus

$$L = \frac{Na}{1 + \exp(-\beta mga)}$$

(c)

Since  $U = mga[N - (L/a)]$ , the force constant is  $mg$ .

12.4

(a)

The possible states are labeled by the number of open links  $n = 0, 1, 2, \dots, N$ . The energy with  $n$  open links is  $E_n = n\Delta$ . The partition function is

$$Q_N = \sum_{n=0}^N e^{-\beta n\Delta} = \frac{1 - e^{-\beta(N+1)\Delta}}{1 - e^{-\beta\Delta}}$$

(b)

The average number of open links is

$$\bar{n} = -\frac{1}{\Delta} \frac{\partial \ln Q_N}{\partial \beta} = \frac{e^{-\beta\Delta}}{1 - e^{-\beta\Delta}} - \frac{(N+1)e^{-\beta(N+1)\Delta}}{1 - e^{-\beta(N+1)\Delta}}$$

The second term is negligible for large  $N$ . At low temperatures  $\beta\Delta \gg 1$  we have

$$\bar{n} \approx e^{-\beta\Delta}$$

12.5

(a)

There are 6 sites in each hexagon, but each site is shared by 3 hexagons. Thus we can assign 2 sites to a hexagon. On the other hand, each hexagon is associated with one interstitial site. Thus, in an infinite lattice, there are half as many interstitial sites as lattice sites.

(b)

The entropy is given by

$$\begin{aligned} \frac{S}{k} &= \ln \Gamma(E) = \ln \Gamma_{\text{vacancy}} + \ln \Gamma_{\text{interstitial}} \\ \Gamma_{\text{vacancy}} &= \frac{N!}{M!(N-M)!} \\ \Gamma_{\text{interstitial}} &= \frac{(N/2)!}{M!(N/2-M)!} \end{aligned}$$

The energy is  $E = M\Delta$ , and the volume fixed, and proportional to  $N$ . Thus, this gives  $S(E, V)$ . Using the Stirling approximation, we obtain

$$\begin{aligned} \frac{S}{k} &= \{N \ln N - M \ln M - (N - M) \ln(N - M)\} + \{N \rightarrow N/2\} \\ &= 2M \ln \frac{N}{\sqrt{2}M} - (N - M) \ln \left(1 - \frac{M}{N}\right) - \left(\frac{N}{2} - M\right) \ln \left(1 - \frac{2M}{N}\right) \end{aligned}$$

(c)

From  $T^{-1} = \partial S / \partial E$  we obtain the relation

$$\left(\frac{E}{N\Delta}\right)^2 = \frac{1}{2} \left(1 - \frac{E}{N\Delta}\right) \left(1 - \frac{2E}{N\Delta}\right) e^{-\Delta/kT}$$

This can be easily solved, but we only give the high- and low-temperature limits:

$$\frac{E}{N\Delta} \approx \begin{cases} 2^{1/2} \exp(-\Delta/2kT) & (kT \gg \Delta) \\ 1/3 & (kT \ll \Delta) \end{cases}$$

The above is equal to the average interstitial fraction  $M/N$  at a given temperature.

12.6

(a)

The partition function for  $N$  non-interacting particles is  $Q_N = Q_1^N$ , where  $Q_1$  is that for a single particle:

$$Q_1 = \sum_{n=1}^3 \exp(-\beta \epsilon_n) = 2e^{-\beta(bx^2 - cx/2)} + e^{-\beta(bx^2 + cx)}$$

The free energy per particle is  $a(x, T) = -kT \ln Q_1$ .

(b)

We find the equilibrium value of  $\bar{x}$  by minimizing  $a(x, T)$  with respect to  $x$ , or maximizing  $Q_1$ . Assume that  $\bar{x}$  is small, and expand the exponential to order  $x^2$ . The condition  $Q'_1 = 0$  gives two roots:

$$\bar{x} = \begin{cases} 0 \\ (4kT/c) [1 - (4bkT/c^2)] \end{cases}$$

Since  $\bar{x}$  cannot be negative, the nontrivial root is acceptable only when  $T < T_c$ , where

$$kT_c = \frac{c^2}{4b}$$

Examining the sign of  $Q''_1$  shows that when  $T < T_c$  the nontrivial roots corresponds to a maximum, while  $\bar{x} = 0$  corresponds to a minimum. For  $T > T_c$ , the only solution is  $\bar{x} = 0$ , which corresponds to a maximum. Thus there is a phase transition at  $T = T_c$ .

12.7

For a classical relativistic gas,

$$Q_N(V, T) = \int \frac{d^3N p d^3N q}{N! h^{3N}} \exp \left[ -\beta \sum_{i=1}^N \sqrt{(cp_i)^2 + (mc^2)^2} \right] = \frac{V^N}{N!} \frac{1}{h^{3N}} I^N(\beta)$$

where

$$I(\beta) = \int d^3p \exp \left[ -\beta \sqrt{(cp)^2 + (mc^2)^2} \right]$$

Using the Sterling approximation to write  $N! \approx N^N$ , we obtain

$$A_N(V, T) = -NkT \left[ \ln \left( \frac{V}{N h^3} \right) + 1 + \ln I(\beta) \right]$$

In the nonrelativistic limit  $kT \ll mc^2$  we have

$$\begin{aligned} I(\beta) &\approx \int d^3p \exp \left[ -\beta \left( mc^2 + \frac{\mathbf{p}^2}{2m} \right) \right] = e^{-mc^2/kT} (2\pi mkT)^{3/2} \\ A_N(V, T) &\approx N \left[ mc^2 + kT \ln \left( \frac{\lambda^3 N}{V} \right) - kT \right] \\ \mu &\approx mc^2 + kT \ln(n\lambda^3) \end{aligned}$$

In ultra-relativistic situations  $kT \gg mc^2$  we can neglect the rest energy, and take

$$\begin{aligned} I(\beta) &\approx \int d^3p e^{-\beta cp} = 4\pi \int_0^\infty dp p^2 e^{-\beta cp} = 8\pi \left( \frac{kT}{c} \right)^3 \\ A_N(V, T) &\approx NkT \left[ \ln \left( \frac{\pi^2 N}{V} \left( \frac{\hbar c}{kT} \right)^3 \right) - 1 \right] \\ \mu &\approx kT \ln(nL^3) \end{aligned}$$

where

$$L = \pi^{2/3} \frac{\hbar c}{kT}$$

12.8

The partition function is  $Q_N = \xi^N$ , where  $\xi$  is the partition function for one particle:

$$\xi = \sum_k e^{-\beta \hbar^2 k^2 / 2m} = \frac{V}{(2\pi)^3} \int d^3k e^{-\beta \hbar^2 k^2 / 2m} = \frac{V}{\lambda^3}$$

where  $\lambda = \sqrt{2\pi \hbar^2 / mkT}$ . The free energy is  $A = -kT \ln Q_N = -NkT \ln(V/\lambda^3)$ . The equation of state is  $P = -\partial A / \partial V = NkT/V$ .

12.9

(a)

The partition is  $Q_N = \xi^N$ , where

$$\xi = \frac{1}{\tau} \int_{-\infty}^{\infty} dp e^{-\beta p^2 / 2m} \int_{-\infty}^{\infty} dq e^{-\beta m \omega^2 q^2 / 2} = \frac{2\pi kT}{\tau \omega}$$

(b)

The free energy is

$$A = -kT \ln Q_N = -NkT \ln \left( \frac{2\pi kT}{\tau\omega} \right)$$

Thus

$$\begin{aligned} S &= -\frac{\partial A}{\partial T} = Nk \left[ 1 + \ln \left( \frac{2\pi kT}{\tau\omega} \right) \right] \\ U &= A + TS = NkT \\ C_V &= \frac{\partial U}{\partial T} + Nk \end{aligned}$$

12.10

(a)

The partition function is  $Q_N = \xi^N$ , where

$$\xi = \sum_{n=0}^{\infty} e^{-\beta \hbar \omega (n+1/2)} = \frac{e^{-\beta \hbar \omega / 2}}{1 - e^{-\beta \hbar \omega}} \xrightarrow{\beta \rightarrow 0} \frac{kT}{\hbar \omega}$$

Comparing with the result of Prob.12.9(a), we find  $\tau = 2\pi\hbar = h$ .

(b)

The state of the  $N$ -oscillator system is labeled by the set of occupation numbers  $\{n_\alpha\}$ , where  $n_\alpha$  is the number of oscillators in state  $\alpha$ , with energy  $\epsilon_\alpha = \hbar\omega(\alpha + 1/2)$ . The possible values of  $\alpha$  are  $0, 1, 2, \dots$ . The occupation numbers have the possible values

$$n_\alpha = \begin{cases} 0, 1 & \text{(fermions)} \\ 0, 1, 2, \dots & \text{(bosons)} \end{cases}$$

They are constraint by the condition

$$\sum_{\alpha} n_{\alpha} = N$$

The energy of a state is given by

$$E\{n_{\alpha}\} = \sum_{\alpha=0}^{\infty} \epsilon_{\alpha} n_{\alpha}$$

The partition function is given by

$$Q_N = \sum_{\{n_{\alpha}\}} e^{-\beta E\{n_{\alpha}\}} = \sum_{\{n_0, n_1, \dots\}} [e^{-\beta \epsilon_0 n_0} e^{-\beta \epsilon_1 n_1} \dots]$$

The sum extends over all possible sets of occupation numbers, subject to the constraint to  $N$  particles. The statistics enters through the possible values of  $n_{\alpha}$ . The evaluation of the partition is difficult because of the constraint.

12.11

(a)

$$Q_N = (e^{\beta \mu_0 B} + e^{-\beta \mu_0 B})^N$$

(b)

$$\langle M \rangle = -\frac{1}{\beta} \frac{\partial}{\partial B} \ln Q_N = \mu_0 N \frac{e^{\beta \mu_0 B} - e^{-\beta \mu_0 B}}{e^{\beta \mu_0 B} + e^{-\beta \mu_0 B}}$$

(c)

$$\langle M^2 \rangle - \langle M \rangle^2 = \frac{1}{\beta^2} \frac{\partial^2}{\partial B^2} \ln Q_N = \frac{4\mu_0^2 N}{(e^{\beta \mu_0 B} + e^{-\beta \mu_0 B})^2}$$

## Chapter 13

13.1

(a)

$$Q(z, T) = \sum_{N=0}^{N_0} \binom{N_0}{N} z^N e^{-\beta N \epsilon} = \sum_{N=0}^{N_0} \binom{N_0}{N} (ze^{-\beta \epsilon})^N = (1 + ze^{-\beta \epsilon})^{N_0}$$

(b)

$$\frac{\langle N \rangle}{N_0} = \frac{z}{N_0} \frac{\partial}{\partial z} \ln Q = \frac{1}{z^{-1} e^{\beta \epsilon} + 1}$$

(c)

$$U = -\frac{\partial}{\partial \beta} \ln Q = \frac{N_0 \epsilon}{z^{-1} e^{\beta \epsilon} + 1}$$

$$C = \frac{\partial U}{\partial T} = \frac{N_0 k (\beta \epsilon)^2 e^{\beta \epsilon}}{z (z^{-1} e^{\beta \epsilon} + 1)^2}$$

13.2

(a)

The grand partition function for the O<sub>2</sub> lattice gas is

$$Q_1(z, T) = \sum_{N_1=0}^N \binom{N}{N_1} (z_1 e^{-\beta \epsilon_1})^{N_1} = (1 + z_1 e^{-\beta \epsilon_1})^N$$

The fraction of occupied sites is

$$\frac{\langle N_1 \rangle}{N_0} = \frac{z_1}{N_0} \frac{\partial}{\partial z_1} \ln Q_1 = \frac{1}{z_1^{-1} e^{\beta \epsilon_1} + 1}$$

Setting the above to  $f = 0.9$ , with  $z_1 = 10^{-5}$  and  $T = 310$  K, we find,

$$\epsilon_1 = kT \ln \frac{z_1(1-f)}{f} \approx -0.37 \text{ eV}$$

(b)

The grand partition function is now given by

$$Q(z, T) = \sum_{N_1=0}^N \sum_{N_2=0}^{N-N_1} \binom{N}{N_1} \binom{N-N_1}{N_2} (z_1 e^{-\beta \epsilon_1})^{N_1} (z_2 e^{-\beta \epsilon_2})^{N_2}$$

$$= \sum_{N_1=0}^N \binom{N}{N_1} (z_1 e^{-\beta \epsilon_1})^{N_1} (1 + z_2 e^{-\beta \epsilon_2})^{N-N_1}$$

$$= (1 + z_1 e^{-\beta \epsilon_1} + z_2 e^{-\beta \epsilon_2})^N$$

The fraction of sites occupied by O<sub>2</sub> is

$$\frac{\langle N_1 \rangle}{N} = \frac{z_1}{N} \frac{\partial}{\partial z_1} \ln Q = \frac{z_1 e^{-\beta \epsilon_1}}{1 + z_1 e^{-\beta \epsilon_1} + z_2 e^{-\beta \epsilon_2}}$$

Set this to 0.1 and solve for  $\epsilon_2$ . With  $\epsilon_1$  from (a), we obtain

$$\epsilon_2 = -0.55 \text{ eV}$$

13.3

(a)

$$E(M) = -\epsilon M$$

$$\Gamma(M) = \binom{N}{M}$$

(b)

The grand partition function of the adsorbed gas is

$$Q(z, T) = \sum_{M=0}^N \binom{N}{M} (ze^{\beta \epsilon})^M = (1 + ze^{\beta \epsilon})^N$$

where  $z = e^{\beta \mu}$ . The average fraction of occupied sites can be obtained either by maximizing the summand using the Stirling approximation:

$$\ln \left[ \binom{N}{M} (ze^{\beta \epsilon})^M \right] \approx M \ln (ze^{\beta \epsilon}) + N \ln N - M \ln M - (N - M) \ln (N - M)$$

or by calculating the grand canonical average:

$$\frac{\bar{M}}{N} = \frac{z}{N} \frac{\partial}{\partial z} \ln Q = \frac{1}{z^{-1} e^{-\beta \epsilon} + 1}$$

(c)

The chemical potential for an ideal gas is given in Prob.3.2:  $\mu = kT \ln (n\lambda^3)$ , where  $\lambda = \sqrt{2\pi\hbar^2/\beta m}$ , and  $n = \beta P$ . In equilibrium, the chemical potential of the adsorbed gas must equal that of the surrounding gas. Thus

$$\frac{\bar{M}}{N} = \frac{\lambda^3 \beta P}{e^{-\beta \epsilon} + \lambda^3 \beta P}$$

(d)

$$\overline{M^2} - \bar{M}^2 = z \frac{\partial}{\partial z} z \frac{\partial}{\partial z} \ln Q = \frac{ze^{-\beta \epsilon}}{(e^{-\beta \epsilon} + z)^2}$$

13.4

(a)

The equation of state is

$$\left(V - \frac{1}{3}\right) \left(P + \frac{3}{V^2}\right) = \frac{8}{3}T$$

Differentiating both sides with respect to  $P$  at constant  $T$ , we find

$$-\frac{\partial V}{\partial P} = \frac{V - \frac{1}{3}}{P + \frac{3}{V^2} - \frac{6}{V^3} \left(V - \frac{1}{3}\right)} = \frac{\left(V - \frac{1}{3}\right)^2}{\frac{8}{3}T - \frac{6}{V^3} \left(V - \frac{1}{3}\right)^2}$$

Near the critical point we put  $V = 1$  and let  $T \rightarrow 1^+$ . Thus

$$\kappa_T = -\frac{1}{V} \frac{\partial V}{\partial P} \approx \frac{1}{6(T-1)}$$

(b) The fractional density fluctuation near the critical point diverges:

$$\frac{\overline{n^2} - \bar{n}^2}{\bar{n}^2} = \frac{T}{V} \kappa_T \approx \frac{1}{6(T-1)}$$

13.5

The condition for equilibrium is  $\ln(n_+ L^3) + \ln(n_- L^3) = 0$ , or  $n_+ n_- = L^{-6}$ . Given  $n_- - n_+ = n_0$ , we find-

$$\begin{aligned} \frac{n_+}{n_0} &= \frac{1}{2} \left[ \sqrt{\frac{4}{(n_0 L^3)^2} + 1} - 1 \right] \\ \frac{n_-}{n_0} &= \frac{1}{2} \left[ \sqrt{\frac{4}{(n_0 L^3)^2} + 1} + 1 \right] \end{aligned}$$

13.6

(a)

Let  $N_i$  be the number of molecules of type  $X_i$  present. The reaction consumes  $\nu_i$  molecules of type  $X_i$ . Thus the change in  $N_i$  is proportional to  $\nu_i$ , with the same proportionality constant for all  $i$ . Hence  $\delta N = \delta N_i / \nu_i$  is independent of  $i$ .

(b)

Minimizing the free energy, we have

$$0 = \delta A = \sum_i \frac{\partial A}{\partial N_i} \delta N_i = \sum_i \mu_i \nu_i \delta N = 0$$

Since  $\delta N$  is arbitrary, we have  $\sum_i \mu_i \nu_i = 0$ .

13.7

(a)

In a fixed volume, the densities obey the relations

$$\frac{\delta n_1}{2} = \delta n_2 = -\frac{\delta n_3}{2}$$

Hence  $A = n_1 - 2n_2$  and  $B = n_1 + n_3$  remain constant.

(b)

The chemical potential for a classical ideal gas is, according to Prob.3.2,  $\mu = kT \ln(n\lambda^3)$ , where  $\lambda = \sqrt{2\pi\hbar^2/mkT}$ . The condition for chemical equilibrium is  $2 \ln(n_1 \lambda_1^3) + \ln(n_2 \lambda_2^3) - 2 \ln(n_3 \lambda_3^3) = 0$  where the  $\lambda_i$  are independent of the densities. Thus

$$\frac{n_1^2 n_2}{n_3^2} = K_0$$

where  $K_0 = (4/9)^3 (mkT/\pi\hbar^2)^{3/2}$ . Two other conditions are

$$\begin{aligned} n_1 + n_3 &= n_0 \\ n_1 &= 2n_2 \end{aligned}$$

These imply

$$n_1^3 = 2K_0 (n_0 - n_1)^2$$

High-temperature limit  $K_0 \rightarrow \infty$ :

$$n_1 \approx n_0 \left(1 - \sqrt{\frac{n_0}{2K_0}}\right)$$

Low-temperature limit  $K_0 \rightarrow 0$

$$n_1 \approx (2K_0 n_0)^{1/3}$$

13.8

(a)

From Sec.8.12 we have

$$b_\ell = \begin{cases} \ell^{-5/2} & \text{(Bose)} \\ (-1)^{\ell+1} \ell^{-5/2} & \text{(Fermi)} \end{cases}$$

(b)

This is a generalization of Prob.11.3. Let  $y = n\lambda^3$ . We seek the expansion

$$\frac{P}{nkT} = \frac{\sum_{\ell=1}^{\infty} b_\ell z^\ell}{\sum_{\ell=1}^{\infty} \ell b_\ell z^\ell} = 1 + a_2 y^2 + a_3 y^3 + \dots$$

by eliminating  $z$  through the relation  $y = \sum_{\ell=1}^{\infty} \ell b_\ell z^\ell$ . To find  $a_2$  and  $a_3$  we rewrite the above as

$$\frac{\sum_{\ell=1}^{\infty} b_\ell z^\ell}{\sum_{\ell=1}^{\infty} \ell b_\ell z^\ell} = 1 + a_2 \left( \sum_{\ell=1}^{\infty} \ell b_\ell z^\ell \right)^2 + a_3 \left( \sum_{\ell=1}^{\infty} \ell b_\ell z^\ell \right)^3 + \dots$$

and expand both sides to order  $z^3$ , obtaining the equation

$$\begin{aligned} & z + b_2 z^2 + b_3 z^3 + \dots \\ &= (z + 2b_2 z^2 + 3b_3 z^3 + \dots) \left[ 1 + a_2 (z + 2b_2 z^2 + \dots)^2 + a_3 (z + \dots)^3 + \dots \right] \\ &= z + (2b_2 + a_2) z^2 + (4b_2 a_2 + a_3 + 3b_3) z^3 + \dots \end{aligned}$$

Equating the coefficients of  $z^2$  and  $z^3$  on both sides, we obtain

$$\begin{aligned} a_2 &= -b_2 \\ a_3 &= 4b_2^2 - 2b_3 \end{aligned}$$

13.9

(a)

Using the result  $\ln Q = \sum_{\lambda} \ln(1 + ze^{-\beta \epsilon_{\lambda}})$  from Sec.13.6, we find in the thermodynamic limit

$$\begin{aligned} \beta P &= a \ln(1 + ze^{\beta B}) + \int \frac{d^3 k}{(2\pi)^3} \ln(1 + ze^{-\beta \epsilon_k}) \\ n &= \frac{a}{z^{-1}e^{-\beta B} + 1} + \int \frac{d^3 k}{(2\pi)^3} \frac{1}{z^{-1}e^{\beta \epsilon_k} + 1} \end{aligned}$$

(b)

For high temperatures  $z \rightarrow 0$ . Thus

$$\begin{aligned} n &\approx zae^{\beta B} + z \int \frac{d^3 k}{(2\pi)^3} e^{-\beta \epsilon_k} = z(ae^{\beta B} + \lambda^{-3}) \approx \frac{z}{\lambda^3} (1 + a\lambda^3) \\ z &\approx n\lambda^3 (1 - a\lambda^3) \end{aligned}$$

where  $\lambda = \sqrt{2\pi\hbar^2/mkT}$ .

(c)

$$\begin{aligned} P &\approx akTze^{\beta B} + zkT \int \frac{d^3 k}{(2\pi)^3} e^{-\beta \epsilon_k} \approx zkT(a + \lambda^{-3}) \\ &\approx n\lambda^3 (1 - a\lambda^3) kT(a + \lambda^{-3}) \approx nkT[1 + O(a^2)] \end{aligned}$$

13.10

$$\begin{aligned} n_b &\approx az \approx na\lambda^3 \\ n_f &\approx \frac{z}{\lambda^3} \approx n(1 - a\lambda^3) \end{aligned}$$

$$\langle n_{\mathbf{k}} \rangle = \frac{1}{Q} \sum_{\{n_1, n_2, \dots\}} n_{\mathbf{k}} \exp[-\beta(\epsilon_1 n_1 + \epsilon_2 n_2 + \dots) + \beta\mu] = -\frac{1}{\beta} \frac{\partial}{\partial \epsilon_{\mathbf{k}}} \ln Q$$

Differentiating both sides with respect to  $\epsilon_{\mathbf{p}}$  with  $\mathbf{p} \neq \mathbf{k}$ , we obtain

$$\begin{aligned} -\frac{1}{\beta} \frac{\partial}{\partial \epsilon_{\mathbf{p}}} \langle n_{\mathbf{k}} \rangle &= \frac{1}{Q} \sum_{\{n_1, n_2, \dots\}} n_{\mathbf{k}} n_{\mathbf{p}} \exp[-\beta(\epsilon_1 n_1 + \epsilon_2 n_2 + \dots) + \beta\mu] \\ &\quad - \frac{1}{\beta} \left( \frac{\partial}{\partial \epsilon_{\mathbf{p}}} \frac{1}{Q} \right) \sum_{\{n_1, n_2, \dots\}} n_{\mathbf{k}} \exp[-\beta(\epsilon_1 n_1 + \epsilon_2 n_2 + \dots) + \beta\mu] \\ &= \langle n_{\mathbf{p}} n_{\mathbf{k}} \rangle - \langle n_{\mathbf{p}} \rangle \langle n_{\mathbf{k}} \rangle \end{aligned}$$

We know that  $\langle n_{\mathbf{k}} \rangle = (z^{-1}e^{\beta \epsilon_k} \pm 1)^{-1}$  does not depend on  $\epsilon_{\mathbf{p}}$ . Thus the above is zero, or

$$\langle n_{\mathbf{p}} n_{\mathbf{k}} \rangle = \langle n_{\mathbf{p}} \rangle \langle n_{\mathbf{k}} \rangle \quad (\mathbf{p} \neq \mathbf{k})$$

13.11

$$\langle \sigma^2 \rangle - \langle \sigma \rangle^2 = \sum_{\mathbf{k} \in G} \sum_{\mathbf{p} \in G} [\langle n_{\mathbf{k}} n_{\mathbf{p}} \rangle - \langle n_{\mathbf{k}} \rangle \langle n_{\mathbf{p}} \rangle] = \sum_{\mathbf{k} \in G} [\langle n_{\mathbf{k}}^2 \rangle - \langle n_{\mathbf{k}} \rangle^2]$$

The last relation follows from the fact that terms with  $\mathbf{k} \neq \mathbf{p}$  do not contribute, as shown in the last problem. By (13.40),  $\langle n_{\mathbf{k}}^2 \rangle - \langle n_{\mathbf{k}} \rangle^2 = \langle n_{\mathbf{k}} \rangle \mp \langle n_{\mathbf{k}} \rangle^2$ . This directly leads to the answer desired.

13.12

(a)

The number of particles in the condensate is  $n_0 = N - N'$ , where  $N' = \sum_{\mathbf{k} \neq 0} n_{\mathbf{k}}$ . Thus

$$\langle n_0^2 \rangle - \langle n_0 \rangle^2 = \sum_{\mathbf{k} \neq 0} [\langle n_{\mathbf{k}}^2 \rangle - \langle n_{\mathbf{k}} \rangle^2] = \sum_{\mathbf{k} \neq 0} \frac{ze^{-\beta \epsilon_k}}{(1 - ze^{-\beta \epsilon_k})^2}$$

(b)

In the infinite-volume limit, when we make the replacement

$$\sum_{\mathbf{k} \neq 0} \rightarrow \frac{V}{(2\pi)^3} \int dk 4\pi k^2$$

we see that the integral diverges at  $z = 1$  due to contributions near  $k = 0$ . For a finite volume, however,  $k$  never assumes the value 0, but has a minimal value of order  $L^{-1} = V^{-1/3}$ . For  $z = 1$ , the integrand near  $k = 0$  is proportional to  $k^{-2}$ , and thus the integral diverges like  $k^{-1} \approx V^{1/3}$ . Thus the fluctuation is proportional to  $V^{4/3}$ .



## Chapter 14

14.1

For given  $a$ , let  $b$  be the length of a diagonal line, and  $\theta$  the angle between a diagonal line and the horizontal. Then  $b \cos \theta = \frac{1}{2}$ , and  $a + 2b \sin \theta = 1$ . These give the relation  $b = \frac{1}{2} \sqrt{1 + (1 - a)^2}$ . The total length of the highway is  $L = 4b + a = 2\sqrt{1 + (1 - a)^2} + a$ . The minimum occurs at  $a = 1 - \frac{1}{\sqrt{3}} = 0.306$ . The minimum length of the highway is  $L = 1 + \sqrt{3} = 2.732$ .

14.2

We wish to solve  $M = \tanh(\beta \gamma J M)$ , where  $\beta = 1/kT$ . Near the transition point  $M$  is small, so we put  $\tanh(x) \approx x - \frac{1}{3}x^3$  and obtain  $M \approx \beta \gamma J M - \frac{1}{3}(\beta \gamma J M)^3$ . This gives

$$M \approx \frac{kT}{(\gamma J)^{3/2}} \sqrt{3(\gamma J - kT)}$$

Thus  $kT_c = \gamma J$ , and  $M \sim \sqrt{T_c - T}$  near the critical point. The critical exponent is therefore  $\beta = \frac{1}{2}$ .

14.3

From (14.59) we have

$$\int d^D x m(x) = \int d^D x |x|^{2-D} e^{-|x|/\xi} = \xi^2 \int d^D y |y|^{2-D} e^{-|y|} \approx t^{-1}$$

The last relation comes from (14.60), which gives  $\xi \approx t^{-1}$ . From (14.30) we have  $\chi \approx t^{-1}$ , with the same critical exponent.

14.4

$$\begin{aligned} m(x) &= \frac{1}{(2\pi)^3} \int_0^\infty dk k^2 2\pi \int_{-1}^1 d(\cos \theta) \frac{e^{ikr \cos \theta}}{k^2 + 2r_0} = \frac{1}{4\pi^2 i r} \int_0^\infty dk k \frac{e^{ikr} - e^{-ikr}}{k^2 + 2r_0} \\ &= -\frac{1}{4\pi^2 r} \frac{\partial}{\partial r} \int_{-\infty}^\infty dk \frac{e^{ikr}}{k^2 + 2r_0} = -\frac{1}{4\pi r} \frac{\partial}{\partial r} \frac{\exp(-r\sqrt{2r_0})}{\sqrt{2r_0}} = \frac{\exp(-r\sqrt{2r_0})}{4\pi r} \end{aligned}$$

The  $dk$  integration was carried out over a contour in the complex plane.

14.5

(a)

Minimize  $E(\eta, \varepsilon)$  with respect to  $\varepsilon$ :

$$\begin{aligned} 0 &= \frac{\partial E}{\partial \varepsilon} = 2c\varepsilon + g\eta^2 \\ \bar{\varepsilon} &= -\frac{g\eta^2}{2c} \end{aligned}$$

(b)

$$\begin{aligned} E_{\text{eff}}(\eta) &= (at - g)\eta^2 + \tilde{b}\eta^4 \\ \tilde{b} &= b - \frac{g^2}{4c} \end{aligned}$$

(c)

Minimize  $E_{\text{eff}}(\eta)$  with respect to  $\eta$ :

$$\begin{aligned} 0 &= \frac{\partial E_{\text{eff}}}{\partial \eta} = \eta [2(at - g) + 4\tilde{b}\eta^2] \\ \bar{\eta} &= \begin{cases} 0 & (t > g/a) \\ \sqrt{(g - at)/2\tilde{b}} & (t < g/a) \end{cases} \end{aligned}$$

The transition temperature corresponds to  $t = g/a$ , or  $T_c + g/a$ . Thus, the effect of the coupling  $g$  is to raise the transition temperature. The equilibrium order parameter in the low-temperature phase is, more explicitly,

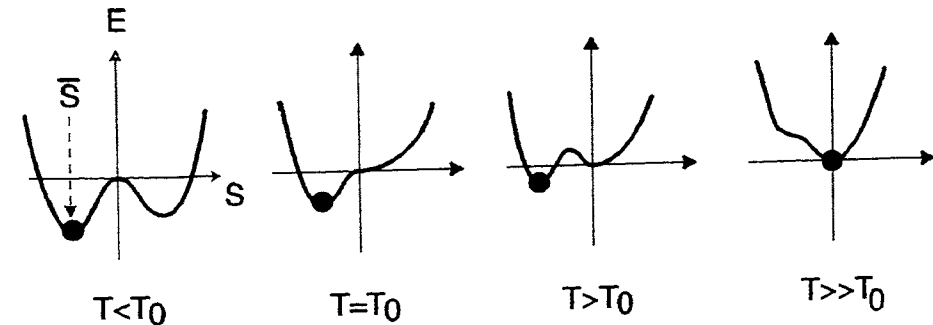
$$\bar{\eta} = \sqrt{\frac{a}{2(b - g^2/4c)}} \sqrt{\left(T_c + \frac{g}{a}\right) - T}$$

which increase as  $g$  increases from 0, and becomes  $\infty$  at  $g = \sqrt{4c}$ . The system becomes unstable for larger values of  $g$ , for  $E_{\text{eff}}$  has no lower bound.

14.6

(a)

If  $b = 0$ , we have the usual quartic curve. Just add a cubic term to get the graphs shown. We see that the transition temperature is greater than  $T_0$ . As we show in more detail later,  $\bar{S}$  jumps abruptly from a finite value to 0, making a first-order phase transition. This is typical of a Landau free energy with a cubic term.



(b)

We see from the sketch above that the conditions for  $E(S)$  to be at a minimum are

- (i)  $\partial E/\partial S = 0$ ,
- (ii)  $E \leq 0$ ,
- (iii)  $\bar{S} \leq 0$ .

The first leads to the condition

$$S(2at + 3bS + 4cS^2) = 0$$

The possible roots are

$$\begin{aligned} S &= 0 \\ S &= \frac{3b}{8c} \left( -1 \pm \sqrt{1 - \frac{32cat}{9b^2}} \right) \end{aligned}$$

For the nontrivial root to correspond to the minimum, it is necessary that  $32cat/9b^2 < 1$ . We see from the graphs that when  $t = 0$  the solution is not  $S = 0$ . Thus we must choose the  $-$  sign. Thus

$$\bar{S} = -\frac{3b}{8c} \left( 1 + \sqrt{1 - \frac{32cat}{9b^2}} \right)$$

if  $32cat < 9b^2$  and  $E(\bar{S}) < 0$ . Otherwise  $\bar{S} = 0$ .

(c)

We note that at  $T = T_c$  we have  $\partial E/\partial S = 0$  and  $E = 0$ . Excluding the trivial root, we can write these two conditions as

$$\begin{aligned} 2at + 3bS + 4cS^2 &= 0 \\ at + bS + cS^2 &= 0 \end{aligned}$$

Multiplying the second equation by 4 and subtracting it from the first, we find  $bS = -2at$ . Substituting this into the second equation gives  $t = b^2/4ac$ , or

$$T_c = T_0 + \frac{b^2}{4ac}$$

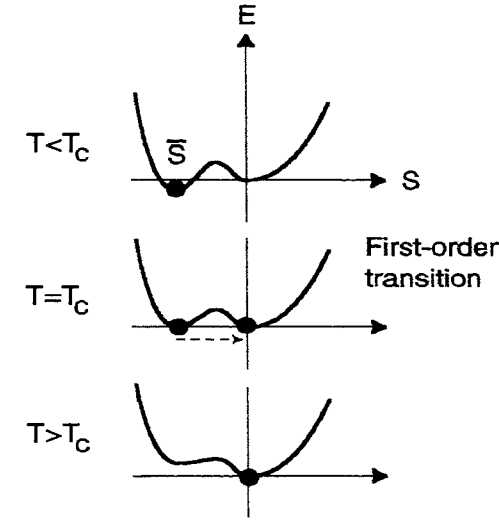
(d)

When the temperature is increase from below to  $T = T_c$ , or  $t = b^2/4ac$ , we have

$$\bar{S} = -\frac{b}{2c}$$

and  $E(\bar{S}) = 0$ . When  $T$  is further decreased it jumps to  $\bar{S} = 0$ , where  $E = 0$ .

Thus the transition is first-order. This is illustrated in the accompanying sketch.



Since  $E$  is the free energy, the entropy of the system is

$$-\frac{\partial E}{\partial T} = -aS^2$$

(It was unthoughtful of us to have used the symbol  $S$  for the order parameter!) Thus, the latent heat is

$$L = aT_c \bar{S}^2 = a \left( T_0 + \frac{b^2}{4ac} \right) \left( \frac{b}{2c} \right)^2$$

(e)

According to the model  $\bar{S}$  decreases steadily as the temperature decreases below  $T_c$ . But the model breaks down when the magnitude of  $\bar{S}$  becomes too large, for it must saturate when molecular alignment becomes perfect.

14.7

(a)

$$\begin{aligned} E &= E_0 + at\phi^2 + \bar{b}\phi^4 \\ \bar{b} &= b + c(\cos^4 \theta + \sin^4 \theta) \end{aligned}$$

Minimize  $E$  with respect to  $\phi$ :

$$\begin{aligned} 0 &= \frac{\partial E}{\partial \phi} = \phi(2at + 4\bar{b}\phi^2) \\ \bar{\phi}^2 &= -\frac{at}{2\bar{b}} \\ E(\bar{\phi}, \theta) &= E_0 - \frac{a^2 t^2}{2\bar{b}} \end{aligned}$$

Thus the minimum of  $E$  occurs at the minimum of  $\bar{b}$ .

(b)

$c < 0$ :  $\theta = 0$ , or  $\theta = \pi/2$ .

$$\bar{b}_{\min} = b - |c|$$

$$\bar{\phi} = \sqrt{\frac{a|t|}{2(\bar{b} - |c|)}}$$

$c = 0$ :  $\theta$  irrelevant.

$$\bar{b}_{\min} = b$$

$$\bar{\phi} = \sqrt{\frac{a|t|}{2b}}$$

$c > 0$ :  $\theta = \pi/4$ .

$$\bar{b}_{\min} = b + \frac{c}{2}$$

$$\bar{\phi} = \sqrt{\frac{a|t|}{2\bar{b} + c}}$$

In all cases, the critical exponent for the order parameter is  $\beta = 1/2$ .

(c)

To find the susceptibility, we turn on an infinitesimally small external field  $\mathbf{h} = (h_1, h_2)$ , so that

$$E = E_0 + at(\phi_1^2 + \phi_2^2) + b(\phi_1^2 + \phi_2^2)^2 - |c|(\phi_1^4 + \phi_2^4) - h_1\phi_1 - h_2\phi_2$$

First consider the case  $\mathbf{h} = (h_1, 0)$ . We can take  $\phi = (\phi_1, 0)$ . The condition  $\partial E / \partial \phi_1 = 0$  gives

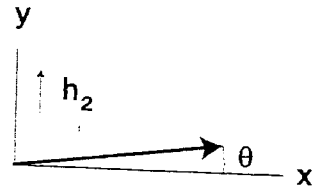
$$2at\phi_1 + 4(b - |c|)\phi_1^3 - h_1 = 0$$

Differentiating with respect to  $h_1$  yields the longitudinal susceptibility:

$$\chi_1 = \frac{\partial \phi_1}{\partial h_1} = \frac{1}{2at + 12(b - |c|)\phi_1^2} = -\frac{1}{3at}$$

The last step is obtained by using  $\phi_1^2 = -at[2(b - |c|)]^{-1}$ . The relevant critical exponent is  $\gamma = 1$ .

Next consider the case  $\mathbf{h} = (0, h_2)$ . We must consider  $\phi = (\phi_1, \phi_2)$  with a small  $\phi_2$  induced by the transverse field. The field causes the order parameter to deviate from the  $x$  axis by an infinitesimal angle  $\theta$ , while the magnitude  $\phi$  is unchanged, as illustrated in the accompanying sketch.



We write  $E$  in the form

$$E = E_0 + at\phi^2 + [b - |c|(\sin^4 \theta + \cos^4 \theta)]\phi^4 - h_2\phi \sin \theta$$

The condition  $\partial E / \partial \phi = 0$  gives

$$2at\phi + 4[b - |c|(\sin^4 \theta + \cos^4 \theta)]\phi^3 - h_2 \sin \theta = 0$$

Differentiating with respect to  $h_2$  and setting  $h_2 = 0$ ,  $\theta = 0$ , we obtain

$$\frac{\partial \theta}{\partial h_2} = \frac{1}{16|c|\phi^3}$$

The transverse susceptibility is

$$\chi_2 = \frac{\partial \phi_2}{\partial h_2} = \phi \frac{\partial \theta}{\partial h_2} = \frac{1}{16|c|\phi^2} = \frac{|c| - b}{16a|c|t}$$

Thus, the associated critical exponent is  $\gamma = 1$ .

## Chapter 15

15.1

Half wavelength of deBroglie wavelength  $\lambda/2 = 1.5 \times 10^{-3} \text{ cm}$ . The wave-number difference between the condensates is  $k_1 - k_2 = \frac{2\pi}{\lambda}$ , and the relative velocity is

$$\frac{\hbar}{m} (k_1 - k_2) = 0.05 \text{ cm/s}$$

15.2

From (15.16)  $|\psi|^2 = g^{-1} (\mu - \frac{1}{2} m \omega^2 r^2)$ . With (15.11), we put  $\mu = g n_0$ , where  $n_0 = N_0/V$ . Thus

$$|\psi|^2 = n_0 \left( 1 - \frac{r^2}{16\pi a n_0 r_0^4} \right)$$

The half-width is  $r_0^2 \sqrt{8\pi a n_0}$ . With the data given, we obtain

$$a = 5.7 \times 10^{-7} \text{ cm}$$

15.3

(a)

From Prob.(10.1), the coordinate of a 1D harmonic oscillator is expressed in terms of creation and annihilation operators through  $q = i\sqrt{\hbar/2m\omega} (a - a^\dagger)$ . Thus

$$x^2 = \frac{\hbar}{m\omega} \left[ a_x^\dagger a_x + \frac{1}{2} - \frac{1}{2} (a_x^2 + a_x^{\dagger 2}) \right]$$

and similarly for  $y$  and  $z$ . The last two terms have no diagonal elements. Therefore

$$\langle \mathbf{n} | x^2 + y^2 + z^2 | \mathbf{n} \rangle = \frac{\hbar}{m\omega} \left( n_x + n_y + n_z + \frac{3}{2} \right)$$

(b)

We know that

$$\left\langle \mathbf{n} \left| \frac{p^2}{2m} + \frac{m\omega^2 r^2}{2} \right| \mathbf{n} \right\rangle = E_{\mathbf{n}}$$

From part (a) we have

$$\frac{m\omega^2}{2} \langle \mathbf{n} | r^2 | \mathbf{n} \rangle = \frac{1}{2} E_{\mathbf{n}}$$

Subtracting the two equations yields the desired result.

(c)

In the  $N$ -boson system, the probability of finding a boson in state  $\mathbf{n}$  is given by the fraction of particles in that state:

$$P(\mathbf{n}) = \frac{1}{N} \frac{1}{z^{-1} \exp(E_{\mathbf{n}}/kT) - 1}$$

Therefore

$$\langle r \rangle^2 = \sum_{\mathbf{n}} \langle \mathbf{n} | r^2 | \mathbf{n} \rangle P(\mathbf{n}) = \frac{r_0^2}{N} \sum_{\mathbf{n}} \frac{(n_x + n_y + n_z + \frac{3}{2})}{z^{-1} \exp(E_{\mathbf{n}}/kT) - 1}$$

15.4

Estimate the transition temperature  $T_0$  by putting the chemical potential equal to the zero-point energy:  $\mu = \frac{3}{2} \hbar \omega$ . Thus

$$\begin{aligned} N &\approx \int_0^\infty dn_x dn_y dn_z \frac{1}{\exp(\hbar \omega (n_x + n_y + n_z)/kT_0) - 1} \\ &= \left( \frac{kT_0}{\hbar \omega} \right)^3 \int_0^\infty dx dy dz \frac{1}{\exp(x + y + z) - 1} \end{aligned}$$

and we obtain

$$\frac{kT_0}{\hbar \omega} = b N^{1/3}$$

where  $b$  is given by

$$b^{-3} = \int_0^\infty dx dy dz \frac{1}{\exp(x + y + z) - 1}$$

15.5

(a)

In the semiclassical approximation

$$\begin{aligned} N &\approx z \int_0^\infty dn_x dn_y dn_z \exp \left( -\beta \hbar \omega \left( n_x + n_y + n_z + \frac{3}{2} \right) \right) \\ &= z (\beta \hbar \omega)^{-3} \exp \left( -\frac{3}{2} \beta \hbar \omega \right) \end{aligned}$$

Thus

$$z \approx N (\beta \hbar \omega)^3 \exp \left( \frac{3}{2} \beta \hbar \omega \right) = \left( \frac{T_0}{bT} \right)^3 \exp \left( \frac{3}{2} \beta \hbar \omega \right)$$

where we have used  $T_0$  defined in Prob.15.4. The chemical potential is

$$\mu \approx kT \ln z = \frac{3}{2} \hbar \omega + 3kT \ln \left( \frac{T_0}{bT} \right)$$

We should put  $b \approx 1$  in the spirit of this approximation, so  $\mu \rightarrow \frac{3}{2} \hbar \omega$  at  $T = T_0$ .

(b)

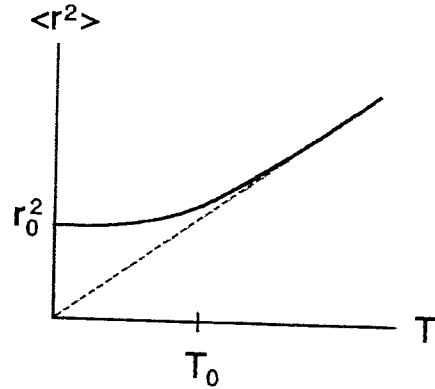
$$\begin{aligned}\frac{\langle r \rangle^2}{r_0^2} &\approx \frac{z}{N} \int dn_x dn_y dn_z \left( n_x + n_y + n_z + \frac{3}{2} \right) \\ &\quad \times \exp \left( -\beta \hbar \omega \left( n_x + n_y + n_z + \frac{3}{2} \right) \right) \\ &= \frac{z}{N} \exp \left( -\frac{3}{2} \beta \hbar \omega \right) \left[ 3 (\beta \hbar \omega)^{-4} + \frac{3}{2} (\beta \hbar \omega)^{-3} \right]\end{aligned}$$

We neglected the second term, and obtain

$$\frac{\langle r \rangle^2}{r_0^2} \approx \frac{3kT}{\hbar \omega} = 3b \left( \frac{T}{T_0} \right) N^{1/3}$$

(c)

According to (b),  $\langle r \rangle^2$  is a linear function of  $T$ . However, the formula becomes invalid below  $T \approx T_0$ , where a condensate begins to form. The condensate wave function is that for  $n_x = n_y = 0$ , and occupies a central region in the harmonic potential of radius  $r_0$ . Thus below  $T_0$  the linear plot flattens to a constant value  $r_0^2$ . (See sketch)



15.6

The equation for the magnetic field is the same as that for the vector potential:

$$\left( \frac{\partial^2}{\partial x^2} - \lambda \right) B(x) = 0$$

where  $\lambda = 0$  outside the medium ( $x < 0$ ), and  $\lambda = 16\pi e^2 n / mc^2$  inside ( $x > 0$ ). Since  $B = B_0$  outside, and  $B(x)$  must be continuous, the solution inside is

$$B(x) = B_0 \exp(-\sqrt{\lambda}x) \quad (x > 0)$$

The penetration depth is  $\lambda^{-1/2}$ .

15.7

(a)

From Ohm's law  $U_0 - V = IR$ . Thus

$$U_0 - \frac{\hbar}{2e} \frac{d\varphi}{dt} = I_0 R \sin \varphi$$

(b)

$$\frac{\hbar}{2e} \frac{d\varphi}{dt} = U_0 (1 - \kappa \sin \varphi)$$

The left side is the voltage. When it approaches zero, we must have  $\sin \varphi = \kappa^{-1}$ . This is possible if  $\kappa > 1$ . Thus there is a nonzero current  $I = \kappa^{-1} I_0$ .

15.8

Verify the solution by substituting it into the equation. The dark soliton has uniform density at infinity  $n = 2b^2/8\pi a$ , which fixes the constant  $b$ .

15.9

(a) The NLSE is

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + g|\psi|^2 \psi$$

The uniform solution  $\psi_0$  is independent of  $x$  with  $|\psi|^2 = n$ . Substituting  $\psi_0 = \sqrt{n} \exp(-i\mu t/\hbar)$  into the equation yields  $\mu = gn$ .

(b)

Putting

$$\psi = (\sqrt{n} + f) \exp(-i\mu t/\hbar)$$

we straightforwardly obtain the first-order equation for  $f$ .

(c)

The complex constants  $U$  and  $V$  satisfy the coupled algebraic equations

$$\begin{aligned}\hbar \omega U &= \frac{\hbar^2 k^2}{2m} U + gnU + gnV \\ -\hbar \omega V &= \frac{\hbar^2 k^2}{2m} V + gnV + gnU\end{aligned}$$

Solving for  $V$  from the second equation and substituting it into the first, we obtain the condition

$$\left( \frac{\hbar^2 k^2}{2m} + gn + \hbar \omega \right) \left( \frac{\hbar^2 k^2}{2m} + gn - \hbar \omega \right) - (gn)^2 = 0$$

which gives the spectrum

$$\omega = \frac{\hbar k}{2m} \sqrt{k^2 + 16\pi a n}$$

(c)

In the long-wavelength limit ( $k \rightarrow 0$ ),  $\omega$  approaches the phonon spectrum  $\omega = ck$  with sound velocity

$$c = \frac{2\hbar}{m} \sqrt{\pi a n}$$

This corresponds to the Goldstone mode. In the short-wavelength limit ( $k \rightarrow \infty$ ) we recover the particle spectrum  $\hbar\omega = \hbar^2 k^2 / 2m$ .

## Chapter 16

16.1

The energy residing in a mode of frequency  $\omega$  of the transmission line is

$$E = \frac{\hbar\omega}{e^{\beta\hbar\omega} - 1} = \frac{\hbar\omega}{\beta\hbar\omega + \frac{1}{2}(\beta\hbar\omega)^2 + \dots} \approx kT \left(1 - \frac{\hbar\omega}{2kT}\right)$$

The second term above gives the first quantum correction. As a estimate use the fundamental mode  $\omega = \pi c/L$ , where  $c$  is the velocity of light, and  $L$  the length of the transmission line. The Nyquist theorem becomes

$$\overline{V^2} = 4kTR\Delta\nu \left(1 - \frac{\pi\hbar c}{2LkT}\right)$$

For  $L = 1$  mm, the correction amounts to approximately 1% at  $T = 300$  K.

16.2

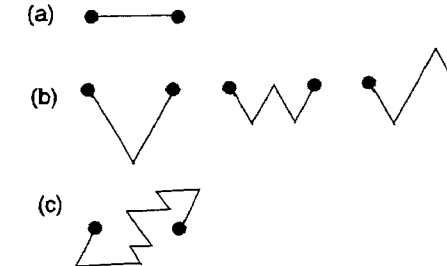
The accompanying sketch illustrates the construction that would lead to a fractal of dimension 2.

(a) Start with a straight line of unit length.

(b) Halve the step size, and double the path length by taking more steps.

The way to do this is not unique. Pick one of the ways.

(c) In the next iteration, each previous segment is independently replaced by a path of twice the length with half the step size.



The path length  $L$  depends on the step size  $\tau$  according to  $L \propto \tau^{1-D}$ , with  $D = 2$ .

16.3

Ignoring the possibility that two suspended particles collide with each other, we can regard the suspension as an ideal gas in equilibrium with the medium, which acts as a heat reservoir. Therefore its partial pressure obeys the ideal gas law.

16.4

It is straightforward to show that

$$n(x, t) = \frac{1}{\sqrt{4\pi Dt}} \exp(-x^2/4Dt)$$

satisfies the diffusion equation. To show the initial condition, note

$$n(x, t) \xrightarrow{t \rightarrow 0} \begin{cases} 0 & (x \neq 0) \\ \infty & (x = 0) \end{cases}$$

and, for all  $t \neq 0$ ,

$$\int_{-\infty}^{\infty} dx n(x, t) = 1$$

Therefore

$$n(x, t) \xrightarrow{t \rightarrow 0} \delta(x)$$

16.5

(a)

For the Brownian particles:  $D = 4 \times 10^{-9} \text{ cm}^2/\text{s}$ .

For  $O_2$ :  $D \approx 0.1 \text{ cm}^2/\text{s}$

Thus an  $O_2$  molecule will travel  $\sqrt{\frac{1}{4} \times 10^{10} \rho} \approx 10 \text{ cm}$ .

(b)

From Einstein's relation  $\eta = D/kT$ .

$F = u/\eta = kTu/D \approx 10^{-5} \text{ dyne}$ .

16.6

Perrin obtained  $A_0 = 7.05 \times 10^{23}$ , which would have led to

$$k = \frac{8.32 \times 10^6}{7.05 \times 10^{23}} = 1.18 \times 10^{-16} \text{ cgs (Modern value: } 1.381 \times 10^{-16})$$

$$e = \frac{2.9 \times 10^{14}}{7.05 \times 10^{23}} = 4.14 \times 10^{-10} \text{ cgs (Modern value: } 4.803 \times 10^{-10})$$

16.7

(a)

Substitute  $\mathbf{j} = -D\nabla n$ , into the continuity equation  $\nabla \cdot \mathbf{j} + \partial n / \partial t = 0$  to obtain

$$-D\nabla^2 n + \frac{\partial n}{\partial t} = 0$$

(b)

With a drift current produced by a uniform constant external force  $\mathbf{F}_{\text{ext}}$ , The total particle current is

$$\mathbf{j} = -D\nabla n + \frac{n}{\eta} \mathbf{F}_{\text{ext}}$$

where  $\eta$  is the mobility. Thus the diffusion equation generalized to

$$-D\nabla^2 n + \frac{1}{\eta} \mathbf{F}_{\text{ext}} \cdot \nabla n + \frac{\partial n}{\partial t} = 0$$

(c)

The absorption, contributes a term  $-V(r)n$  to the rate of change of the particle density. From this point of view, the Schrödinger equation describes a diffusion in imaginary time, with absorption, of the wave function  $\psi$ . What makes quantum mechanics distinctive is that  $\psi$  is a complex probability amplitude, and not a probability.

## Chapter 17

17.1

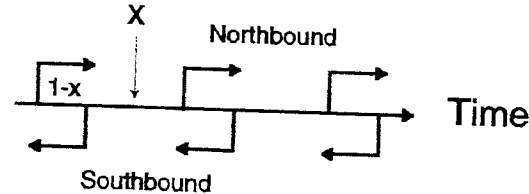
If the showers are distributed at random, the probability that one occurred on Tuesday would be  $1/7$ , and the probability that it did not occur would be  $6/7$ . The probability that none of the 12 showers occur on Tuesday would be  $(6/7)^{12} = 0.157$ . Better bring the umbrella.

17.2

If parking tickets were issued at random, the probability of getting 12 tickets on two days of the week would be  $(2/7)^{12} = 3 \times 10^{-7}$ . This is so small that we must reject the assumption that tickets were given out at random, and advise the student to use a parking lot on those days. Of course, this assumes that the police maintains the same tactic.

17.3

What determines whether the man goes north or south is the correlation between northbound and southbound trains, as illustrated in the sketch. If he enters the station during the interval  $x$ , he goes north. Otherwise he goes south. Since he went north 70% of the time, we conclude  $x = 0.7$ .

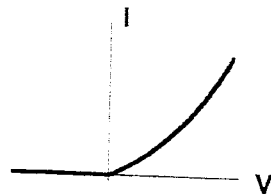


17.4

Generate a sequence of random number between 0 and 1. Divide the interval (0,1) into say 10 equal bins, and keep a running score of the number of random numbers in each bin, as they are being generated. At the end of the run, plot a histogram of the numbers in each bin. If the sequence is truly random, the histogram should fit a Poisson distribution.

17.5

The current-voltage characteristic of the device is shown in the accompanying sketch.



Let the probability of finding the voltage to have a value between  $V$  and  $V + dV$  be  $P(V)dV$ .

Let the probability of finding the current to have a value between  $I$  and  $I + dI$  be  $Q(I)dI$ .

The current is never negative. So  $Q(I) = 0$  for  $I < 0$ . For  $I > 0$ , we have

$$Q(I) = P(V) \frac{dV}{dI} = P(V) \frac{V_0}{I + I_0} \quad (I > 0)$$

where

$$V = V_0 \ln \left( 1 + \frac{I}{I_0} \right)$$

In the range  $V \leq 0$ , we must have

$$\int_{-\infty}^0 dI Q(I) = \int_{-\infty}^0 dV P(V) \equiv \alpha$$

Thus  $Q(I)$  should contain a term  $\alpha \delta(I)$ . The complete result is

$$Q(I) = \begin{cases} P(V) \frac{V_0}{I + I_0} & (I \geq 0) \\ \alpha \delta(I) & (I < 0) \end{cases}$$

17.6

Let the probability density for  $y$  be  $Q(y)$ . We have

$$\int_{-\infty}^y dy' Q(y') = \int_{-\infty}^x dx' P(x')$$

where  $x = \sqrt{y}/b$ . The integrands on both sides are zero for negative arguments.

Thus

$$\int_0^y dy' Q(y') = \int_0^x dx' \frac{x'}{a} \exp \left( -\frac{x'^2}{2a} \right) = \exp \left( -\frac{y}{2ab} \right) - 1$$

Differentiating this gives

$$Q(y) = \frac{1}{2ab} \exp \left( -\frac{y}{2ab} \right) \quad (y \geq 0)$$



## Chapter 18

18.1

(a)

$$\begin{aligned} S(\omega) &= \int_{-\infty}^{\infty} d\tau e^{i\omega\tau} G(\tau) \\ The &= \nu \int_{-\infty}^{\infty} d\tau e^{i\omega\tau} \int_{-\infty}^{\infty} dt f(t) f(t+\tau) + 2\pi\delta(\omega) I^2 \\ &= \nu |f_{\omega}^2| + 2\pi\delta(\omega) I^2 \end{aligned}$$

where  $I = \nu \int_{-\infty}^{\infty} dt f(t) I = \nu \int_{-\infty}^{\infty} dt f(t)$  by Campbell's theorem (17.29)..

(b)

$$\begin{aligned} f_{\omega} &= \int_0^{\infty} dt e^{i\omega t - \lambda t} = (i\omega - \lambda)^{-1} \\ I &= \nu \int_0^{\infty} dt e^{-\lambda t} = \nu/\lambda. \end{aligned}$$

$$S(\omega) = \frac{\nu}{\omega^2 + \lambda^2} + \frac{2\pi\nu^2}{\lambda^2} \delta(\omega)$$

The white-noise component is  $\nu/\lambda^2$ , the first term in the limit  $\lambda \gg \omega$ .

18.2

We can use the result of Prob.18.1(a), with

$$f_{\omega} = \int_{-\infty}^{\infty} dt e^{i\omega t} \varphi(t) \xrightarrow{\omega \rightarrow 0} \int_{-\infty}^{\infty} dt \varphi(t) \equiv q.$$

$$I = \nu \int_{-\infty}^{\infty} dt \varphi(t) = \nu q.$$

$$S(\omega) \xrightarrow{\omega \rightarrow 0} \nu q + 2\pi\nu^2 q^2 \delta(\omega)$$

18.3

(a)

$I(t)$  and  $I(t+\tau)$  are the same if there are an even number of sign changes during  $\tau$  and equal but opposite if there are an odd number of sign changes. Thus

$$\langle I(t)I(t+\tau) \rangle = a^2 (P_{\text{even}} - P_{\text{odd}})$$

(b)

The probability that there are  $k$  crossing in the time interval  $\tau > 0$  is given by the Poisson distribution

$$P(k; \nu) = \frac{(\nu\tau)^k}{k!} e^{-\nu\tau}$$

The probability there are an even and odd number of crossings are given respectively by

$$\begin{aligned} P_{\text{even}} &= e^{-\nu\tau} \left( 1 + \frac{(\nu\tau)^2}{2!} + \frac{(\nu\tau)^4}{4!} + \dots \right) \\ P_{\text{odd}} &= e^{-\nu\tau} \left( \nu\tau + \frac{(\nu\tau)^3}{3!} + \frac{(\nu\tau)^5}{5!} + \dots \right) \end{aligned}$$

Thus for  $\tau > 0$

$$\langle I(t)I(t+\tau) \rangle = a^2 e^{-\nu\tau} \left( 1 - \nu\tau + \frac{(\nu\tau)^2}{2!} - \frac{(\nu\tau)^3}{3!} + \dots \right) = a^2 e^{-2\nu\tau}$$

If  $\tau < 0$ , then  $\langle I(t)I(t+\tau) \rangle = \langle I(t-|\tau|)I(t) \rangle = \langle I(t)I(t+|\tau|) \rangle$  by invariance in time translation. Thus the general answer is obtained by replacing  $\tau$  by  $|\tau|$ .

(c)

$$\begin{aligned} S(\omega) &= \int_{-\infty}^{\infty} d\tau \langle I(t)I(t+\tau) \rangle = a^2 \int_0^{\infty} d\tau e^{i\omega\tau - \nu\tau} + a^2 \int_{-\infty}^0 d\tau e^{i\omega\tau + \nu\tau} \\ &= 2a^2 \text{Re} \int_0^{\infty} d\tau e^{i\omega\tau - \nu\tau} = \frac{2\nu a^2}{\omega^2 + \nu^2} \end{aligned}$$

18.4

From (18.2)

$$W_3(3, 1, 2) = \int dx_4 W_4(3, 1, 4, 2)$$

From (18.21) and (18.26)

$$\begin{aligned} W_3(3, 1, 2) &= W_2(3, 1)P(3, 1|2) = W_2(3, 1)P(1|2) \\ W_4(3, 1, 4, 2) &= W_3(3, 1, 4)P(3, 1, 4|2) = W_3(3, 1, 4)P(4|2) \\ &= W_2(3, 1)P(1|4)P(4|2) \end{aligned}$$

Substituting these into the last equation we obtain

$$\begin{aligned} W_2(3, 1)P(1|2) &= \int dx_4 W_2(3, 1)P(1|4)P(4|2) \\ P(1|2) &= \int dx_4 P(1|4)P(4|2) \end{aligned}$$

18.5

(a)

The diffusion equation is invariant under space-time translations. The solution in Sec.16.5 was based on the choice  $x = 0$  and  $t = 0$  for the space-time origin. We can generalize the result by replacing  $x, t$  by  $x - x_0, t - t_0$ , respectively.

(b)

Denote

$$P(i|j) = \frac{1}{\sqrt{4\pi D(t_i - t_j)}} \exp\left(-\frac{(x_i - x_j)^2}{4D(t_i - t_j)}\right)$$

First show, for  $n = 2$ ,

$$P(2|0) = \int dx_1 P(2|1)P(1|0)$$

The right side is

$$X = \frac{1}{4\pi D \sqrt{(t_2 - t_1)(t_1 - t_0)}} \int_{-\infty}^{\infty} dx_1 \exp\left(-\frac{1}{4D} \left( \frac{(x_2 - x_1)^2}{t_2 - t_1} - \frac{(x_1 - x_0)^2}{t_1 - t_0} \right)\right)$$

The exponent can be written as

$$-\frac{1}{4D} \left( \frac{x_2^2}{t_2 - t_1} + \frac{x_0^2}{t_1 - t_0} \right) - \frac{A}{4D} \left( x_1^2 + \frac{2B}{A} x_1 \right)$$

where

$$\begin{aligned} A &= \frac{1}{t_2 - t_1} + \frac{1}{t_1 - t_0} \\ B &= \frac{x_2}{t_2 - t_1} + \frac{x_0}{t_1 - t_0} \end{aligned}$$

Perform the integral, we obtain

$$\begin{aligned} X &= \frac{\sqrt{4\pi D/A}}{4\pi D \sqrt{(t_2 - t_1)(t_1 - t_0)}} \exp\left(-\frac{1}{4D} \left( \frac{(x_2 - x_1)^2}{t_2 - t_1} - \frac{(x_1 - x_0)^2}{t_1 - t_0} - \frac{B^2}{A} \right)\right) \\ &= \frac{1}{\sqrt{4\pi D(t_2 - t_0)}} \exp\left(-\frac{(x_2 - x_0)^2}{4D(t_2 - t_0)}\right) \end{aligned}$$

Next show that the result for  $n - 1$  implies that for  $n$ , where  $n > 2$ . The integrals one has to do is similar to the one above.

18.6

The instructions are fairly explicit.

18.7

Compute the thermal average of the total magnetic moment  $M$ , as a function of temperature  $T$ . It is easy to obtain a rough value for  $T_c$ , but it is difficult to attain precision, for the transition will not be sharp, owing to the finite lattice

size. When one increases the lattice size, the system will take longer and longer to attain thermal equilibrium near the critical point, because it takes longer and longer for large blocks to flip due to thermal fluctuations. This difficulty is known as "critical slowing down", and is of course the essence of spontaneous symmetry breaking. To overcome this, one has to improve the algorithm by making trial flips of not just single spins, but blocks of spins of random sizes.