Kaluza-Klein Gravity

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PHYS 569

December 11, 2008

Abstract

We review Kaluza's five-dimensional theory of gravity and discuss the means by which the seemingly unnatural cylinder hypothesis can be removed. Along the way we see that the geometry of a five dimensional empty universe induces four dimensional electromagnetic radiation (and implies Maxwell's equations) when the cylinder condition is assumed and induces four dimensional massive matter when the cylinder condition is dropped.

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1 Introduction

Kaluza attempted to unify Einstein's theory of gravity with Maxwell's theory of electromagnetism by adding a fifth dimension to the universe. Kaluza assumed that there would be no five-dimensional matter, that the mathematical structure of general relativity would be extended to five dimensions without changes, and that the quantities would have no dependence on the fifth coordinate. A consequence of his work was that an empty five-dimensional universe implied the presence of four-dimensional electromagnetic radiation and Maxwell's equations. However, the third assumption, called the cylinder condition, was considered unnatural. Klein and others attempted to modify Kaluza's theory in a manner that allowed the cylinder condition to be dropped. Some of this work led to modern string theory as one possible grand unification theory.

In this paper, we will review Kaluza's theory and discuss directions in which the cylinder condition has been removed. The direction depends in part on how one interprets the fifth dimension. Klein assumed the fifth dimension was lengthlike. This required that the fifth dimension was compact. This leads to theories in which higher-dimensional matter must be present. The STM hypothesis (where "STM" stands for "Space-Time-Matter"), on the other hand, assumes the fifth dimension is masslike. This leads to non-compactified five dimensional theories of gravity in which, when one removes the cylinder condition, four dimensional massive matter is induced by the geometry of a five-dimensional empty universe.

2 Maxwell's Equations and Relativity

2.1 Maxwell's Equations in Special Relativity

In order to understand Kaluza's initial results, we must begin with Maxwell's equations. Maxwell's equations are the fundamental equations in electricity and magnetism. Electromagnetic radiation is a consequence of these equations, as is the value of the speed of light. Maxwell's equations in their traditional differential form are as follows:

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \tag{2.1}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{2.2}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{2.3}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}. \tag{2.4}$$

The vectors $\vec{E}$ and $\vec{B}$ are the electric and magnetic fields respectively, ρ is charge density, $\vec{J}$ is the current density, and μ_0 and ϵ_0 are constants related to the speed of light by

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}.$$

Equations (2.1) and (2.2) are Gauss' law for the electric and magnetic fields respectively. Equation (2.3) is Faraday's law, and equation (2.4) is the Ampère-Maxwell law.

We will rewrite Maxwell's equations in a form that is suitable for special relativity. To do so, we define the basic rank one and rank two tensors that we will need. First we define the 4-current to be:

$$\mathbf{J} = (c\rho, \vec{J}).$$

There is a continuity equation given by

$$\partial_\mu J^\mu = 0.$$

This is interpreted physically as a local conservation of charge. This implies the familiar continuity equation of electricity and magnetism:

$$\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t}.$$

Next we define the 4-potential. The electric field $\vec{E}$ is conservative and is therefore the gradient of a scalar potential Φ. Equation (2.2) indicates that $\vec{B}$ is a solenoidal vector field, and hence is the curl of a vector field $\vec{A}$, called the vector potential. We define the 4-potential in the obvious way:

$$\mathbf{A} = (\Phi/c, \vec{A}).$$

Next define the electromagnetic field tensor to be the rank two antisymmetric contravariant tensor $\mathbf{F}$ whose components are

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu.$$

Note that the partial derivative operator with an upper index in the context of a general metric is given by

$$\partial^\mu = g^{\alpha\mu}\partial_\alpha.$$

If we arrange the components of $\mathbf{F}$ in a matrix, we have

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_1/c & E_2/c & E_3/c \\ -E_1/c & 0 & B_3 & -B_2 \\ -E_2/c & -B_3 & 0 & B_1 \\ -E_3/c & B_2 & -B_1 & 0 \end{pmatrix}.$$

Recall that the classical Lorentz force from the electric and magnetic fields on a charge q moving at velocity $\vec{u}$ is given by

$$\vec{f} = q\vec{E} + q\vec{u} \times \vec{B}.$$

The 4-force can be shown to satisfy

$$f^\mu = q u_\nu F^{\mu\nu}$$

where u_ν are the covariant components of the 4-velocity of the charge. See pages 458-459 of [PS02] for the derivation. Finally, we define the dual field tensor $\mathbf{G}$ to be an antisymmetric contravariant rank 2 tensor whose components are

$$G^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta},$$

where $F_{\alpha\beta}$ is $\mathbf{F}$ with its indices lowered and $\epsilon^{\mu\nu\alpha\beta}$ is a generalization of the rank 3 Levi-Cevita tensor. The components of $\mathbf{G}$ are given by

$$G^{\mu\nu} = \begin{pmatrix} 0 & B_1 & B_2 & B_3 \\ -B_x & 0 & -E_3/c & E_2/c \\ -B_2 & E_3/c & 0 & -E_1/c \\ -B_3 & -E_2/c & E_1/c & 0 \end{pmatrix}.$$

In terms of these tensors, Maxwell's equations can be written

$$\partial_\nu F^{\mu\nu} = \mu_0 J^\mu \tag{2.5}$$
$$\partial_\nu G^{\mu\nu} = 0. \tag{2.6}$$

See pages 461-462 of [PS02].

As formulated, Maxwell's equations are invariant under Lorentz boosts. See section 12.4 of [PS02] for the argument.

2.2 Maxwell's Equations in General Relativity

When passing to more general coordinates, we want to write Maxwell's equations in a covariant form appropriate for general relativity. We cannot merely replace the partial derivatives with covariant derivatives. Doing so would cause conservation laws to be violated. See Section 4.3 of [Wal84] for a discussion of this issue. The correct way to write Maxwell's equations in general coordinates is

$$\nabla^\mu F_{\mu\nu} = \kappa J_\nu \tag{2.7}$$
$$\nabla_\lambda F_{\mu\nu} - \nabla_\mu F_{\lambda\nu} + \nabla_\nu F_{\lambda\mu} = 0, \tag{2.8}$$

where κ is a constant. In terms of the 4-potential, in order to preserve conservation laws we introduce a term involving the Ricci curvature, giving us

$$\nabla^\mu \nabla_\mu A_\nu - R^\lambda{}_\nu A_\lambda = \kappa J_\nu.$$

See section 4.3 of [Wal84].

3 Kaluza's Five Dimensional Spacetime

Kaluza brought gravity and electromagnetism together with a five-dimensional theory of spacetime. The key assumptions of Kaluza's model are:

1. the model should maintain Einstein's vision that nature is pure gravity;
2. the mathematics of general relativity is not altered, merely extended to five dimensions; and
3. there is no dependence on the fifth coordinate.

The last assumption, called the "cylinder assumption," appears to be the most contrived. However, we will see it is possible to interpret the fifth coordinate in a manner where this assumption is more natural.

In accordance with the second assumption, the definition of the Christoffel symbols, the Riemann tensor, the Ricci tensor, the Einstein tensor, and other fundamental quantities of general relativity are not altered. They are merely extended so that the indices run from 0 to 4. Such indices will be denoted by capital letters. Assumption 1 implies that the initial equation should be free of matter. Therefore the Einstein equation in five dimensions is

$$G_{AB} = 0$$

where G is the five-dimensional Einstein tensor. As in the four-dimensional case, this implies that

$$R_{AB} = 0 \tag{3.1}$$

where R is the five-dimensional Ricci tensor. As in the four dimensional case, everything reduces to the properties of the metric.

Consider a four dimensional metric tensor $g_{\alpha\beta}$. Kaluza extended this metric to force the fifth dimension to induce electromagnetism. To do so, the $g_{\alpha 4}$ components of the metric are connected with the electromagnetic potential $A_{\alpha\beta}$ and g_{44} is defined in terms of a scalar field φ. There is also a scaling constant κ, necessary for certain results when this is treated variationally. The four dimensional metric is taken to have signature $(+\ -\ -\ -)$. We work in units where $c = 1, \hbar = 1$ and $G = 1$. The five-dimensional metric given by Kaluza, arranged as a matrix in block form, is:

$$g_{AB} = \begin{pmatrix} g_{\alpha\beta} + \kappa^2\varphi^2 A_\alpha A_\beta & \kappa\varphi^2 A_\alpha \\ \kappa\varphi^2 A_\beta & \varphi^2 \end{pmatrix}.$$

Now we assume the cylinder conditions: all derivatives (covariant or otherwise) with respect to the 4-coordinate are zero. Let

$$T^{EM}_{\alpha\beta} = \frac{1}{4} g_{\alpha\beta} F_{\gamma\delta} F^{\gamma\delta} - F^{\gamma}{}_{\alpha} F_{\beta\gamma}$$

be the electromagnetic stress-energy tensor, $G_{\alpha\beta}$ the 4-dimensional Einstein tensor, and $\Box$ the D'Alembertion operator $g^{\alpha\beta}\partial_\beta\partial_\alpha$. With the cylinder condition, plugging the metric g_{AB} into equation (3.1) yields the field equations:

$$\begin{aligned} G_{\alpha\beta} &= \frac{\kappa^2\varphi^2}{2} T^{EM}_{\alpha\beta} - \frac{1}{\varphi}\left[\nabla_\alpha(\partial_\beta\varphi) - g_{\alpha\beta}\Box\varphi\right], && (3.2) \\ \nabla^\alpha F_{\alpha\beta} &= -3\frac{\partial^\alpha\varphi}{\varphi} F_{\alpha\beta}, \text{ and} && (3.3) \\ \Box\varphi &= \frac{\kappa^2\varphi^3}{4} F_{\alpha\beta}F^{\alpha\beta}. && (3.4) \end{aligned}$$

Now assume the scalar field φ is constant. Let the scaling parameter be given so that

$$\kappa\varphi = 4\sqrt{\pi G}$$

where we include G only for purposes of recognizing our results as something more familiar. Then equations (3.2) and (3.3) become:

$$\begin{aligned} G_{\alpha\beta} &= 8\pi G\varphi^2 T^{EM}_{\alpha\beta} && (3.5) \\ \nabla^\alpha F_{\alpha\beta} &= 0. && (3.6) \end{aligned}$$

This is Kaluza and Klein's original result. Equation (3.5) is the Einstein equation and equation (3.6) is the same as Equation (2.7) in the absence of a current. Therefore, the Einstein field equation with no matter in five dimensions gives two of Maxwell's equations in general coordinates.[1] On the other hand, if φ is constant then equation (3.4) becomes

$$F_{\alpha\beta}F^{\alpha\beta} = 0.$$

This is a little more problematic, and raises some of the early controversy surrounding the presence of a cosmological constant.

There are two consequences of equations (3.5) and (3.6). First of all, Maxwell's

[1]It is not clear whether or not Kaluza and Klein obtained Equation (2.8).

equations are now part of the field equations that are obtained from the five-dimensional Einstein equation in a vacuum. In other words, Maxwell's equations, and as a consequence electromagnetism, are a product of pure geometry. The other consequence is that the Einstein equation in a vacuum in five dimensions induces the Einstein equation *with* matter (electromagnetic radiation) in four dimensions. Therefore matter in the observable universe is a consequence of geometry in a five-dimensional universe.

4 Dropping the Cylinder Condition

The consequences of Kaluza's theory led to several attempts to fix its shortcomings. These shortcomings were twofold. The first, and most obvious, problem is that a fifth dimension is not observed. The second problem is that the cylinder condition seems unnatural. If one assumes that the fifth dimension is lengthlike, the typical way the first of these shortcoming is resolved involves the assumption that the extra dimension is compact. In this setting, the cylinder condition is abandoned and the consequences are studied. Another way to resolve these shortcomings is to assume that the fifth dimension represents something other than time or space. This leads to noncompactified theories. For more detail on these theories than that presented below, see [OW97] and the citations therein.

4.1 Compactified Theories

Klein modified Kaluza's theory on the assumption that the fifth dimension was lengthlike. This in turn requires the fifth dimension to be very "small." In other words, the fifth dimension had to be compact with a small diameter. Specifically, Klein assumed that the fifth dimension was a circle, that is topologically $\mathbb{S}^1$, with a very small radius.[2] As of the end of the twentieth century, observations constrained the radius of the circles in this dimension to be less than 10^{-18} meters [OW97] (citing [KS91]).

Next one drops the cylinder condition. Since the fifth dimension is circular, the dependence on the fifth coordinate is periodic. Consequently metric

[2]Since we are assuming the fifth dimension should be a smooth manifold, by the classification of 1-manifolds, the assumption that the fifth dimension is topologically $\mathbb{S}^1$ is equivalent to the assumption that the fifth dimension is compact.

coefficients and components of other tensors can be expanded into Fourier series:

$$\begin{aligned} g_{\alpha\beta}(x,y) &= \sum_{n\in\mathbb{Z}} g_{\alpha\beta}^{(n)} \mathrm{e}^{iny/r} \\ A_\alpha(x,y) &= \sum_{n\in\mathbb{Z}} A_\alpha^{(n)}(x) \mathrm{e}^{iny/r} \\ \varphi(x,y) &= \sum_{n\in\mathbb{Z}} \varphi^{(n)}(x) \mathrm{e}^{iny/r} \end{aligned}$$

where r is the radius of the circles in the fifth dimension, x is a vector corresponding to position in four dimensional spacetime, y is a coordinate in the fifth dimension, and the superscript (n) denotes the n^{th} Fourier mode.

The expansion of the fundamental quantities into Fourier series above hints at an explanation of the quantization of charge. Unfortunately this would involve consequences for masses of various modes which diverge from observation.

It is also tempting to add other "compact" dimensions to bring the strong and weak forces into the game. The number of required dimensions is typically placed at ten or eleven. The number of dimensions should be the smallest that can accommodate the minimum symmetry group in the standard model: $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$. At least six of these dimensions are compact, consisting of Calabi-Yau 3-folds (complex manifolds of 3 dimensions have 6 real dimensions). One problem, however, is that matter must be present in the additional dimensions, violating Kaluza's first assumption.

4.2 Non-Compactified Theories

If one does not assume that the fifth dimension is lengthlike, and one does not assume the cylinder condition, there is no reason to assume the fifth dimension is compact and no restrictions on its 1-dimensional topology are necessary. One alternative is to associate the fifth dimension with rest mass. This is called the STM-hypothesis (where "STM" stands for "Space-Time-Matter"). There are several reasons why this interpretation is theoretically appealing. See section 6.10 of [OW97].

Without the cylinder condition, we may assume we are in coordinates in which the electromagnetic potential vanishes. We also drop the scaling factor κ. The metric tensor is now given by

$$\hat{g}_{AB} = \begin{pmatrix} g_{\alpha\beta} & 0 \\ 0 & \epsilon\varphi \end{pmatrix}$$

where we put hats on components of five-dimensional tensors to avoid confusion. Here ϵ is ± 1, depending on the appropriate sign in the signature of the metric for the fifth dimension.

The $\alpha\beta$ components of the Ricci tensor are now given by:

$$\hat{R}_{\alpha\beta} = R_{\alpha\beta} - \frac{\nabla_\beta(\partial_\alpha\varphi)}{\varphi} + \frac{\epsilon}{2\varphi^2}\left(\frac{\partial_4\varphi\partial_4 g_{\alpha\beta}}{\varphi} - \partial_4 g_{\alpha\beta} + g^{\gamma\delta}\partial_4 g_{\alpha\gamma}\partial_4 g_{\beta\delta} - \frac{g^{\gamma\delta}\partial_4 g_{\alpha\beta}}{2}\right). \tag{4.1}$$

There are also equations for the other components of the Ricci tensor. See section 6.2 of [OW97], but we will not discuss them in this paper.

We assume Kaluza's first hypothesis. Therefore the Einstein equation reduces to

$$\hat{R}_{AB} = 0.$$

This implies:

$$R_{\alpha\beta} = \frac{\nabla_\beta(\partial_\alpha\varphi)}{\varphi} - \frac{\epsilon}{2\varphi^2}\left(\frac{\partial_4\varphi\partial_4 g_{\alpha\beta}}{\varphi} - \partial_4 g_{\alpha\beta} + g^{\gamma\delta}\partial_4 g_{\alpha\gamma}\partial_4 g_{\beta\delta} - \frac{g^{\gamma\delta}\partial_4 g_{\alpha\beta}}{2}\right). \tag{4.2}$$

We would like to show that this assumption on the geometry induces the existence of matter in four dimensions. We require the existence of a matter stress-energy tensor $T_{\alpha\beta}$ such that

$$8\pi G T_{\alpha\beta} = G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta}.$$

Contracting equation (4.2) with $g^{\alpha\beta}$ (and invoking an equation following from the 44-component of $\hat{R}_{AB}$) yields the four-dimensional scalar curvature

$$R = \frac{\epsilon}{4\varphi^2}\left[\partial_4 g^{\alpha\beta}\partial_4 g_{\alpha\beta} + (g^{\alpha\beta}\partial_4 g_{\alpha\beta})^2\right].$$

Hence the 4-dimensional Einstein equation can be written

$$
\begin{aligned}
8\pi G T_{\alpha\beta} &= \frac{\nabla_\beta(\partial_\alpha\varphi)}{\varphi} - \frac{\epsilon}{2\varphi^2}\left[\frac{\partial_4\varphi\partial_4 g_{\alpha\beta}}{\varphi} - \partial_4 g_{\alpha\beta} + g^{\gamma\delta}\partial_4 g_{\alpha\gamma}\partial_4 g_{\beta\delta} - \right. \\
&\left. \frac{g^{\gamma\delta}\partial_4 g_{\alpha\beta}}{2} + \frac{g_{\alpha\beta}}{4}\left(\partial_4 g^{\alpha\beta}\partial_4 g_{\alpha\beta} + (g^{\alpha\beta}\partial_4 g_{\alpha\beta})^2\right)\right].
\end{aligned}
$$

We can use this equation to *define* the matter stress-energy tensor. If defined as such, $T_{\alpha\beta}$ is symmetric. In addition, if we reintroduce the cylinder condition, then contracting $T_{\alpha\beta}$ with the metric yields

$$T = g^{\alpha\beta}T_{\alpha\beta} = 0.$$

This in turn implies that the "matter fluid" satisfies an equation of state $p = \rho/3$. This implies radiation, which suggests the result obtained originally by Kaluza. That is, in the presence of the cylinder condition, five dimensional geometry induces 4-dimensional matter that takes on the form of electromagnetic radiation (i.e. photons). Removing the cylinder condition allows five dimensional geometry to induce four-dimensional massive matter.

There are several solutions of the field equation (4.2) and the field equations that come out of expressions for the components R_{A4} of the Ricci tensor. See sections 6.4 through 6.8 of [OW97].

5 Conclusion

We have seen that adding a fifth dimension and assuming the cylinder condition, the five dimensional Einstein field equations in a vacuum imply the presence of 4-dimensional electromagnetic radiation and Maxwell's equations. If one assumes that the fifth dimension is lengthlike, it is natural to also assume that the fifth dimension is compact and very small. This can be extended to higher dimensions in an attempt to create a grand unified theory. However the presence of higher-dimensional matter is typically required. On the other hand, one can interpret the fifth dimension to be masslike. Then the fifth dimension need not be compact. As a consequence, one can define the four-dimensional mass stress-energy tensor in such a way that the five-dimensional Einstein field equations without the cylinder condition induces the presence of massive matter in four dimensions. In other words, the presence of matter is a consequence of the geometry of the universe.

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